Signals as Vectors
Systems as Maps

ELEC 3004: Systems: Signals & Controls
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Lecture 3

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Follow Along Reading:

- Chapter 1: 
  *Introduction to Signals and Systems*
  - § 1.7 Classification of Systems

- Chapter 3: 
  *Signal Representation By Fourier Series*
  - § 3.1 Signals and Vectors
  - § 3.3 Signal Representation by Orthogonal Signal Set

Linearity (Superposition) Recap
(from Lecture 2)
Linear Systems: Superposition

- **Given** input $x_1(t)$ produces output $y_1(t)$ and input $x_2(t)$ produces output $y_2(t)$

- **Then**: The linearly combined input
  \[ x(t) = ax_1(t) + bx_2(t) \]
  must produce the linearly combined output
  \[ y(t) = ay_1(t) + by_2(t) \]
  for arbitrary $a$ and $b$

- **Generalizing**:
  - Input: $x(t) = \sum_k a_k x_k(t)$ | Output: $y(t) = \sum_k a_k y_k(t)$

Linear Systems: Superposition Consequences

**Consequences:**

- Zero input for all time yields a zero output.
  - This follows readily by setting $a = 0$, then $0 \cdot x(t) = 0$

- For an invertible system,
  Zero output for all time means yields that there was zero input

- DC output/Bias $\rightarrow$ **Incrementally linear**
  - Ex: $y(t) = [2x(t)] + [1]$
  - Set offset to be added offset [Ex: $y_0(t)=1$]
Example: Is it Linear?

\[ C \frac{dV_o(t)}{dt} = V_i(t) - V_o(t) \]

\[ C \left( V_O(s) \cdot s \right) = \left( \frac{1}{R} \right) \left( V_i(s) - V_O(s) \right) \]

\[ (RCs + 1) V_O(s) = V_i(s) \]

\[ V_O(s) = \left( \frac{1}{RCs + 1} \right) V_i(s) \]

\[ \therefore \text{ (Therefore)}: \]

- Voltage (or current) superposition may be employed 😊
- Zero output, means Zero input (?)

Example: Is it First-Order?

If \( V_i = 0 \):

\[ [aY(s) \cdot s + bY(s) = 0] \]

\[ \frac{aY(s)}{s} + bY = 0 \]

- “Autonomous System”
- Natural (or unforced) response
- \( RC \) response \( V_O(s) - V_O(0) + V_O(0) = 0 \)
- \( T = a/b = RC \)
- Time Constant: \( \tau = \frac{1}{T} = \frac{1}{RC} \)
- Solution: \( V_O(t) = [V_O(0)] e^{-\frac{t}{RC}} \)
Signals as Vectors

Complex Exponential Signals

\[ x(t) = Ae^{\lambda t} \]

- \( A \) and \( \lambda \) are generally complex numbers.

- If \( A \) and \( \lambda \) are, in fact, real-valued numbers, \( x(t) \) is itself real-valued and is called a **real exponential**
Signals as Vectors

• Back to the beginning!

\[ F(x) \]

\[ F(\ldots) = \text{system} \]

6 March 2019 - ELEC 3004: Systems

• There is a perfect analogy between signals and vectors …

\textbf{Signals are vectors!}

• A vector can be represented as a sum of its components in a variety of ways, depending upon the choice of coordinate system. A signal can also be represented as a sum of its components in a variety of ways.
Signals as Vectors

- Represent them as Column Vectors

\[ x = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ \vdots \\ x[N] \end{bmatrix} . \]

Signals as Vectors

- Can represent phenomena of interest in terms of signals

- Natural vector space structure (addition/subtraction/norms)

- Can use norms to describe and quantify properties of signals
Signals as vectors

Signals can take real or complex values.
In both cases, a natural vector space structure:

- Can add two signals: \[ x_1[n] + x_2[n] \]
- Can multiply a signal by a scalar number: \[ C \cdot x[n] \]
- Form linear combinations: \[ C_1 \cdot x_1[n] + C_2 \cdot x_2[n] \]

Various Types

- Audio signal (sound pressure on microphone)
- B/W video signal (light intensity on photosensor)
- Voltage/current in a circuit (measure with multimeter)
- Car speed (from tachometer)
- Robot arm position (from rotary encoder)
- Daily prices of books / air tickets / stocks
- Hourly glucose level in blood (from glucose monitor)
- Heart rate (from heart rate sensor)
Vector Refresher

- Length: 
  \[ |x|^2 = x \cdot x \]

- Decomposition: 
  \[ x = c_1 y + e_1 = c_2 y + e_2 \]

- Dot Product of \( \perp \) is 0: 
  \[ x \cdot y = 0 \]

Vectors [2]

- Magnitude and Direction
  \[ f \cdot x = |f| |x| \cos(\theta) \]

- Component (projection) of a vector along another vector
  \[ f = cx + e \quad \text{← Error Vector} \]
Vectors [3]

• \( \infty \) bases given \( \vec{x} \)

![Diagram of vectors](image.png)

• Which is the best one?

\[
\begin{align*}
f &= \vec{e}_x \\
e|\vec{x}| &= |\vec{f}| \cos \theta \\
e|\vec{x}|^2 &= |\vec{f}| |\vec{x}| \cos \theta = \vec{f} \cdot \vec{x} \\
e &= \frac{\vec{f} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} = \frac{1}{|\vec{x}|^2} \vec{f} \cdot \vec{x} \\
\vec{f} \cdot \vec{x} &= 0
\end{align*}
\]

• Can I allow more basis vectors than I have dimensions?

Signals **Are** Vectors

• A Vector / Signal can represent a sum of its components

  Remember (Lecture 5, Slide 10):
  Total response = Zero-input response + Zero-state response

  ![Response Diagram](image.png)

• Vectors are Linear
  – They have **additivity** and **homogeneity**
**Vectors / Signals Can Be Multidimensional**

- A signal is a quantity that varies as a function of an index set

- They can be multidimensional:
  - 1-dim, discrete index (time): \( x[n] \)
  - 1-dim, continuous index (time): \( x(t) \)
  - 2-dim, discrete (e.g., a B/W or RGB image): \( x[j; k] \)
  - 3-dim, video signal (e.g, video): \( x[j; k; n] \)

**It’s Just a Linear Map**

- \( y[n] = 2u[n-1] \) is a linear map
- BUT \( y[n] = 2(u[n]-1) \) is **NOT** Why?

- **Because of homogeneity!**
  \[ T(au) = aT(u) \]
Norms of signals

Can introduce a notion of signals being "nearby."

This is characterized by a metric (or distance function).

\[ d(x, y) \]

If compatible with the vector space structure, we have a norm.

\[ \| x - y \| \]

Examples of Norms

Can use many different norms, depending on what we want to do.

The following are particularly important:

- \( \ell_2 \) (Euclidean) norm:

\[ \| x \|_2 = \left( \sum_{k=1}^{n} |x[k]|^2 \right)^{\frac{1}{2}} \quad \text{norm}(x, 2) \]

- \( \ell_1 \) norm:

\[ \| x \|_1 = \sum_{k=1}^{n} |x[k]| \quad \text{norm}(x, 1) \]

- \( \ell_\infty \) norm:

\[ \| x \|_\infty = \max_k |x[k]| \quad \text{norm}(x, \infty) \]

What are the differences?
Properties of norms

For any norm \( \| \cdot \| \), and any signal \( x \), we have:

- **Linearity**: if \( C \) is a scalar,
  \[
  \| C \cdot x \| = |C| \cdot \| x \|
  \]

- **Subadditivity (triangle inequality)**:
  \[
  \| x + y \| \leq \| x \| + \| y \|
  \]

Can use norms:

- To detect whether a signal is (approximately) zero.
- To compare two signals, and determine if they are "close."

\[
\| x - y \| \approx 0
\]

Signal representation by **Orthogonal Signal Set**

- **Orthogonal Vector Space**

\[\Rightarrow\] A signal may be thought of as having components.
Linear combinations of signals

Application Example: Active Noise Cancellation

A “noise” signal, that we want to get rid of.

- At subject location, signal is
  \[ x[n] \]

- Microphone picks up signal
  \[ x_c[n] \]

- Subtract the two signals:
  \[ y(t) = x(t) - x_c(t) \]

Notice careful synchronization is needed!
Component of a Signal

\[ f(t) \approx c x(t) \quad t_1 \leq t \leq t_2 \]

\[ c = \frac{\int_{t_1}^{t_2} f(t) x(t) \, dt}{\int_{t_1}^{t_2} x^2(t) \, dt} = \frac{1}{E_x} \int_{t_1}^{t_2} f(t) x(t) \, dt \]

- Let’s take an example:

\[ f(t) \approx c \sin t \quad 0 \leq t \leq 2\pi \]

\[ x(t) = \sin t \quad \text{and} \quad E_x = \int_0^{2\pi} \sin^2(t) \, dt = \pi \]

![Graph](image)

Thus

\[ f(t) \approx \frac{4}{\pi} \sin t \quad (3.14) \]

Basis Spaces of a Signal

\[
\int_{t_1}^{t_2} x_m(t) x_n(t) \, dt = \begin{cases} 
0 & m \neq n \\
E_n & m = n 
\end{cases}
\]

\[ f(t) \approx c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t) \]

\[ = \sum_{n=1}^{N} c_n x_n(t) \]

\[ e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t) \]

\[ c_n = \frac{\int_{t_1}^{t_2} f(t) x_n(t) \, dt}{\int_{t_1}^{t_2} x_n^2(t) \, dt} \]

\[ = \frac{1}{E_n} \int_{t_1}^{t_2} f(t) x_n(t) \, dt \quad n = 1, 2, \ldots, N \]

\[ f(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t) + \cdots \]

\[ = \sum_{n=1}^{\infty} c_n x_n(t) \quad t_1 \leq t \leq t_2 \]
Observe that the error energy $Ee$ generally decreases as $N$, the number of terms, is increased because the term $C_k^2 E_k$ is nonnegative. Hence, it is possible that the error Energy $\rightarrow 0$ as $N \rightarrow \infty$. When this happens, the orthogonal signal set is said to be complete.

In this case, it’s no more an approximation but an equality

---

A fundamental idea of linear algebra

One basis maybe better suited for a particular problem

For vectors $w_1, \ldots, w_n$ to be a basis for $\mathbb{R}^n$, this means:

1. The $w_i$ ’s are linearly independent
2. A $n \times n$ matrix $W$ with these columns is invertible
3. Every vector $v$ in $\mathbb{R}^n$ can be written in exactly one was as a combination of the $w_i$ ’s

$$v = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n$$
BREAK

Systems as Maps
Then a System is a **MATRIX**

\[
\begin{bmatrix}
  y[1] \\
  y[2] \\
  \vdots \\
  y[M]
\end{bmatrix}
= 
\begin{bmatrix}
  D_{11} & D_{12} & \cdots & D_{1N} \\
  D_{21} & D_{22} & \cdots & D_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  D_{M1} & D_{M2} & \cdots & D_{MN}
\end{bmatrix}
\begin{bmatrix}
  u[1] \\
  u[2] \\
  \vdots \\
  u[N]
\end{bmatrix}.
\]

\[
y[i] = \sum_j D_{ij} u[j].
\]

Linear Time Invariant

- Linear & Time-invariant (of course - tautology!)
- Impulse response: \( h(t) = F(\delta(t)) \)
- Why?
  - Since it is linear the output response \( y(t) \) to any input \( x(t) \) is:
    \[
    x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau
    \]
    \[
    y(t) = F\left[ \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau \right] \xrightarrow{linear} \int_{-\infty}^{\infty} x(\tau) F[\delta(t-\tau)] \, d\tau
    \]
    \[
    h(t-\tau) \xrightarrow{LT} F[\delta(t-\tau)]
    \]
    \[
    \Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau = x(t) * h(t)
    \]
- The output of any continuous-time LTI system is the **convolution** of input \( u(t) \) with the impulse response \( F(\delta(t)) \) of the system.
Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

\[
a_0 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}
\]

Laplace:

\[
a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)
\]

\[
A(s)Y(s) = B(s)X(s)
\]

• Total response = Zero-input response + Zero-state response

Initial conditions
External Input

Linear Systems and ODE’s

• Linear system described by differential equation

\[
a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}
\]

• Which using Laplace Transforms can be written as

\[
a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)
\]

\[
A(s)Y(s) = B(s)X(s)
\]

where \(A(s)\) and \(B(s)\) are polynomials in \(s\)
Unit Impulse Response

- $\delta(t)$: Impulsive excitation
- $h(t)$: characteristic mode terms

Ex:

EXAMPLE 2.4

Determine the unit impulse response $h(t)$ for a system specified by the equation:

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0$$

This is a second order system (2 DoF) having the characteristic polynomial:

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

The characteristic roots of this system are $\lambda_1 = -1$ and $\lambda_2 = -2$. Therefore:

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

Differentiation of this equation yields:

$$y(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

The initial conditions are (see Eq. (2.26) for $t = 0^+$):

$$y(0^+) = y(0) = 0$$

Setting $t = 0^+$ in Eq. (2.26a) and (2.26b) and substituting the initial conditions just given, we obtain:

$$y_0 = c_1 + c_2$$

$$1 = -c_1 - 2c_2$$

Solving these simultaneous equations yields:

$$c_1 = \frac{1}{3}$$
$$c_2 = -\frac{2}{3}$$

Therefore:

$$y(t) = \frac{e^{-t}}{3} - \frac{2e^{-2t}}{3}$$

More Examples ☺
(Elaborating from Lecture 2)
Another 2\textsuperscript{nd} Order System: Accelerometer or Mass Spring Damper (MSD)

- General accelerometer:
  - Linear spring (\(k\)) (0\textsuperscript{th} order w/r/t o)
  - Viscous damper (\(b\)) (1\textsuperscript{st} order)
  - Proof mass (\(m\)) (2\textsuperscript{nd} order)

\(\Rightarrow\) Electrical system analogy:
- resistor (R) : damper (b)
- inductance (L) : spring (k)
- capacitance (C) : mass (m)

Measuring Acceleration: Sense \(\mathbf{a}\) by measuring spring motion \(\mathbf{Z}\)

- Start with Newton’s 2\textsuperscript{nd} Law:
  \[ m\mathbf{a} = \mathbf{F} \]

- Substitute:
  \[ m\frac{d^2\mathbf{x}}{dt^2} = k(\mathbf{X} - \mathbf{x}) + b\frac{d(\mathbf{X} - \mathbf{x})}{dt} \]

\[ \mathbf{Z} \equiv (\mathbf{X} - \mathbf{x}) \rightarrow \mathbf{x} = \mathbf{X} - \mathbf{Z} \]

\[ \Rightarrow m\frac{d^2\mathbf{Z}}{dt^2} = m\frac{d^2\mathbf{Z}}{dt^2} + k\mathbf{Z} + b\frac{d\mathbf{Z}}{dt} \]

- Solve ODE:
  \[ \mathbf{X}(t) = X_0e^{i\omega t} \quad \mathbf{Z}(t) = Z_0e^{i\omega t} \]
Measuring Acceleration [2]

- Substitute candidate solutions:

\[ m \frac{d^2(X_0 e^{i\omega t})}{dt^2} = m \frac{d^2(Z_0 e^{i\omega t})}{dt^2} + k \left( Z_0 e^{i\omega t} \right) + b \frac{d(Z_0 e^{i\omega t})}{dt} \]

\[ -m \omega^2 X_0 e^{i\omega t} = -m \omega^2 Z_0 e^{i\omega t} + k Z_0 e^{i\omega t} + (i\omega) b Z_0 e^{i\omega t} \]

- Define Natural Frequency \( (\omega_0) \) & Simplify for \( Z_0 \)

(the spring displacement “magnitude”):

\[
\omega_0 \equiv \sqrt{\frac{k}{m}}
\]

\[
Z_0 = \frac{m \omega^2 X_0}{m \omega^2 - k - i\omega b} = \frac{X_0}{\sqrt{1 - \frac{\omega_0^2}{\omega^2} - \frac{b^2}{m^2 \omega^2}}}
\]

Acceleration: 2nd Order System

- For \( \omega << \omega_0 \):

\[
Z_0 \approx \frac{\omega^2 X_0}{\omega_0^2} = \frac{a}{\omega_0^2}
\]

\[
\rightarrow a = Z_0 \omega_0^2
\]

\( \rightarrow \) it’s an Accelerometer

- For \( \omega \sim \omega_0 \)

  – As: \( b \rightarrow 0 \), \( Z \rightarrow \infty \)
  – Sensitivity ↑

- For \( \omega >> \omega_0 \):

\[
Z_0 \approx X_0
\]

\( \rightarrow \) it’s a Seismometer
Cascades of Linear Systems:
Ex. 6: Quarter-Car Model

Example: Quarter-Car Model (2)

\[ \dot{x} + \frac{b}{m_1} (\dot{x} - \dot{y}) + \frac{k_s}{m_1} (x - y) + \frac{k_w}{m_1} x = \frac{k_w}{m_1} r, \]

\[ \dot{y} + \frac{b}{m_2} (\dot{y} - \dot{x}) + \frac{k_s}{m_2} (y - x) = 0. \]

\[ s^2 X(s) + s \left( X(s) - Y(s) \right) + \frac{k_s}{s} \left( X(s) - Y(s) \right) + \frac{k_w}{s} X(s) = \frac{k_w}{s} R(s), \]

\[ s^2 Y(s) + \frac{b}{m_2} \left( Y(s) - X(s) \right) + \frac{k_s}{m_2} \left( Y(s) - X(s) \right) = 0, \]

\[ \frac{Y(s)}{R(s)} = \frac{k_w b}{s^4 + \left( \frac{b}{m_1} + \frac{b}{m_2} \right) s^3 + \left( \frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1} \right) s^2 + \left( \frac{k_w b}{m_1 m_2} \right) s + \frac{k_w k_s}{m_1 m_2}. \]
Next Time...

• We’ll talk about Other System Properties 😊

• We will introduce this via the lens of:
  “Systems as Maps. Signals as Vectors”

• Review:
  – Phasers, complex numbers, polar to rectangular, and general functional forms.
  – Chapter B and Chapter 1 of Lathi
    (particularly the first sections on signals & classification thereof)

• Register on Platypus

• Try the practise assignment