Signals as Vectors
Systems as Maps

ELEC 3004: Systems: Signals & Controls
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Lecture 3

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Tomorrow: UN International Women's Day 2017

• Ada Lovelace: English mathematician and writer
• Creator of the first algorithm and first computer program
Lecture Schedule:

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Follow Along Reading:

- **Chapter 1:** *Introduction to Signals and Systems*
  - § 1.7 Classification of Systems

- **Chapter 3:** *Signal Representation By Fourier Series*
  - § 3.1 Signals and Vectors
  - § 3.3 Signal Representation by Orthogonal Signal Set
## System Terminology

### System Classifications/Attributes

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems
Dynamical Systems...

- A system with a memory
  - Where past history (or derivative states) are **relevant** in determining the response
- Ex:
  - RC circuit: Dynamical
    - Clearly a function of the “capacitor’s past” (initial state) and
    - Time! (charge / discharge)
  - R circuit: is memoryless ∵ the output of the system
    (recall V=IR) at some time \( t \) only depends on the input at time \( t \)

- Lumped/Distributed
  - Lumped: Parameter is constant through the process & can be treated as a “point” in space
  - Distributed: System dimensions ≠ small over signal
    - Ex: waveguides, antennas, microwave tubes, etc.

Causality:
Causal (physical or nonanticipative) systems

- Is one for which the output at any instant \( t_0 \) depends only on the value of the input \( x(t) \) for \( t \leq t_0 \). Ex:
  \[
  u(t) = x(t-2) \Rightarrow \text{causal} \quad u(t) = x(t-2) + x(t+2) \Rightarrow \text{noncausal}
  \]
- A “real-time” system must be causals
  - How can it respond to future inputs?
- A prophetic system: knows future inputs and acts on it (now)
  - The output would begin before \( t_0 \)
- In some cases Noncausal maybe modelled as causal with delay
- Noncausal systems provide an upper bound on the performance of causal systems
**Causality:**
Looking at this from the output’s perspective...

- **Causal** = The output before some time $t$ does not depend on the input after time $t$.

Given: $y(t) = F(u(t))$

For:

$\tilde{u}(t) = u(t), \forall 0 \leq t < T \text{ or } [0, T)$

Then for a $T > 0$:

$\rightarrow \tilde{y}(t) = y(t), \forall 0 \leq t < T$

### Systems with Memory

- A system is said to have memory if the output at an arbitrary time $t = t_*$ depends on input values other than, or in addition to, $x(t_*)$

- Ex: Ohm’s Law

$$V(t_o) = Ri(t_o)$$

- **Not** Ex: Capacitor

$$V(t_0) = \frac{1}{C} \int_{-\infty}^{t} i(t) \, dt$$
**Time-Invariant Systems**

- **Given** a shift (delay or advance) in the input signal
- **Then/Causes** simply a like shift in the output signal

- If \( x(t) \) produces output \( y(t) \)
- Then \( x(t - t_0) \) produces output \( y(t - t_0) \)

- Ex: Capacitor
  - \( V(t_0) = \frac{1}{C} \int_{-\infty}^{t} i(\tau - t_0) \, d\tau \)
  - \( = \frac{1}{C} \int_{t_0}^{t} i(\tau) \, d\tau \)
  - \( = V(t - t_0) \)

![Diagram of time-invariant system](image-url)
Unit Step Function

- \( u(t) = \begin{cases} 
0, & t < 0 \\
1, & t > 0 
\end{cases} \)

“Rectangular Pulse”

- \( p(t) = u(t) - u(t - T) \)

Unit-Impulse Function

1. \( \delta(t) = 0 \) for \( t \neq 0 \).
2. \( \delta(t) \) undefined for \( t = 0 \).
3. \( \int_{t_1}^{t_2} \delta(t) \, dt = \begin{cases} 
1, & \text{if } t_1 < 0 < t_2 \\
0, & \text{otherwise} 
\end{cases} \)
EXAMPLE: First Order RC Filter

- Passive, First-Order Resistor-Capacitor Design:

\[
\begin{align*}
\text{(Low-pass configuration)} \\
\end{align*}
\]

- 3dB (½ Signal Power):
  \[
  \omega = 2\pi f
  \]
  \[
  f_c = \frac{1}{2\pi RC}
  \]

- Magnitude:
  \[
  |V_{\text{out}}| = \sqrt{\frac{1}{(\omega RC)^2}} |V_{\text{in}}|
  \]

- Phase:
  \[
  \phi = \tan^{-1} (-\omega RC')
  \]

Example 1: RC Circuits

\[
\begin{align*}
y(t) &= R f(t) + \frac{1}{C} \int_{-\infty}^{t} f(\tau) \, d\tau \\
y(t) &= R f(t) + \frac{1}{C} \int_{0}^{t} f(\tau) \, d\tau + \frac{1}{C} \int_{0}^{t} f(\tau) \, d\tau \\
y(t) &= v_C(0) + R f(t) + \frac{1}{C} \int_{0}^{t} f(\tau) \, d\tau \\
y(t) &= v_C(t_0) + R f(t) + \frac{1}{C} \int_{t_0}^{t} f(\tau) \, d\tau
\end{align*}
\]
BREAK

Signals as Vectors
Complex Exponential Signals

\[ x(t) = A e^{\lambda t} \]

- \( A \) and \( \lambda \) are generally complex numbers.

- If \( A \) and \( \lambda \) are, in fact, real-valued numbers, \( x(t) \) is itself real-valued and is called a real exponential.

![Diagram](image)

Signals as Vectors

- Back to the beginning!
There is a perfect analogy between signals and vectors …

**Signals are vectors!**

A vector can be represented as a sum of its components in a variety of ways, depending upon the choice of coordinate system. A signal can also be represented as a sum of its components in a variety of ways.

Represent them as Column Vectors

\[ x = \begin{bmatrix} 
  x[1] \\
  x[2] \\
  x[3] \\
  \vdots \\
  x[N] 
\end{bmatrix}. \]
Signals as Vectors

• Can represent phenomena of interest in terms of signals

• Natural vector space structure (addition/subtraction/norms)

• Can use norms to describe and quantify properties of signals

Signals as vectors

Signals can take real or complex values.

In both cases, a natural vector space structure:

- Can add two signals: \( x_1[n] + x_2[n] \)
- Can multiply a signal by a scalar number: \( C \cdot x[n] \)
- Form linear combinations: \( C_1 \cdot x_1[n] + C_2 \cdot x_2[n] \)
Various Types

- Audio signal (sound pressure on microphone)
- B/W video signal (light intensity on photosensor)
- Voltage/current in a circuit (measure with multimeter)
- Car speed (from tachometer)
- Robot arm position (from rotary encoder)
- Daily prices of books / air tickets / stocks
- Hourly glucose level in blood (from glucose monitor)
- Heart rate (from heart rate sensor)

Vector Refresher

- Length: $|x|^2 = x \cdot x$
- Decomposition: $x = c_1 y_1 + c_2 y_2$
- Dot Product of $\perp$ is 0: $x \cdot y = 0$
Vectors [2]

- Magnitude and Direction

\[ f \cdot x = |f||x| \cos(\theta) \]

- Component (projection) of a vector along another vector

\[ f = cx + e \quad \text{Error Vector} \]

Vectors [3]

- \( \infty \) bases given \( \vec{x} \)

- Which is the best one?

\[ f = cx \]
\[ c|x| = |f| \cos \theta \]
\[ c|x|^2 = |f||x| \cos \theta = f \cdot x \]
\[ c = \frac{f \cdot x}{x \cdot x} = \frac{1}{|x|^2} f \cdot x \]
\[ f \cdot x = 0 \]

- Can I allow more basis vectors than I have dimensions?
Signals Are Vectors

- A Vector / Signal can represent a sum of its components

  Remember (Lecture 5, Slide 10):
  \[ \text{Total response} = \text{Zero-input response} + \text{Zero-state response} \]

- Vectors are Linear
  - They have additivity and homogeneity

Vectors / Signals Can Be Multidimensional

- A signal is a quantity that varies as a function of an index set

- They can be multidimensional:
  - 1-dim, discrete index (time): \( x[n] \)
  - 1-dim, continuous index (time): \( x(t) \)
  - 2-dim, discrete (e.g., a B/W or RGB image): \( x[j; k] \)
  - 3-dim, video signal (e.g, video): \( x[j; k; n] \)
It's Just a Linear Map

- $y[n] = 2u[n] - 1$ is a linear map
- BUT $y[n] = 2(u[n] - 1)$ is NOT Why?

- Because of homogeneity!
  
  $T(au) = aT(u)$

Norms of signals

Can introduce a notion of signals being "nearby."

This is characterized by a **metric** (or distance function).

$$d(x, y)$$

If compatible with the vector space structure, we have a **norm**.

$$\|x - y\|$$
Examples of Norms

Can use many different norms, depending on what we want to do. The following are particularly important:

- \( \ell_2 \) (Euclidean) norm:

\[
\|x\|_2 = \left( \sum_{k=1}^{n} |x[k]|^2 \right)^{\frac{1}{2}} \quad \text{norm}(x, 2)
\]

- \( \ell_1 \) norm:

\[
\|x\|_1 = \sum_{k=1}^{n} |x[k]| \quad \text{norm}(x, 1)
\]

- \( \ell_\infty \) norm:

\[
\|x\|_\infty = \max_k |x[k]| \quad \text{norm}(x, \infty)
\]

What are the differences?

Properties of norms

For any norm \( \| \cdot \| \), and any signal \( x \), we have:

- Linearity: if \( C \) is a scalar,

\[
\|C \cdot x\| = |C| \cdot \|x\|
\]

- Subadditivity (triangle inequality):

\[
\|x + y\| \leq \|x\| + \|y\|
\]

Can use norms:

- To detect whether a signal is (approximately) zero.
- To compare two signals, and determine if they are “close.”

\[
\|x - y\| \approx 0
\]
Signal representation by Orthogonal Signal Set

- Orthogonal Vector Space

A signal may be thought of as having components.

Component of a Signal

\[ f(t) = c \pi(t) \quad t_1 \leq t \leq t_2 \]

\[ c = \frac{\int_{t_1}^{t_2} f(t) \pi(t) \, dt}{\int_{t_1}^{t_2} \pi^2(t) \, dt} = \frac{1}{E_\pi} \int_{t_1}^{t_2} f(t) \pi(t) \, dt \]

- Let’s take an example:

\[ f(t) = c \sin t \quad 0 \leq t \leq 2\pi \]

\[ \pi(t) = \sin t \quad \text{and} \quad E_\pi = \int_0^{2\pi} \sin^2(t) \, dt = \pi \]

\[ f(t) = c \sin t \]

Fig. 3.5 Approximation of square signal in terms of a single sinusoid.

Thus

\[ f(t) = \frac{4}{\pi} \sin t \quad (3.14) \]
Basis Spaces of a Signal

$$\int_{t_1}^{t_2} x_m(t)x_n(t) \, dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

\[ f(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t) \]
\[ = \sum_{n=1}^{N} c_n x_n(t) \]

\[ e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t) \]

\[ c_n = \frac{\int_{t_1}^{t_2} f(t)x_n(t) \, dt}{\int_{t_1}^{t_2} x_n^2(t) \, dt} \]
\[ = \frac{1}{E_n} \int_{t_1}^{t_2} f(t)x_n(t) \, dt \quad n = 1, 2, \ldots, N \]

\[ f(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t) + \cdots \]
\[ = \sum_{n=1}^{\infty} c_n x_n(t) \quad t_1 \leq t \leq t_2 \]

- Observe that the error energy $E_e$ generally decreases as $N$, the number of terms, is increased because the term $Ck^2 E_k$ is nonnegative. Hence, it is possible that the error energy $\to 0$ as $N \to \infty$. When this happens, the orthogonal signal set is said to be complete.
- In this case, it’s no more an approximation but an equality
Linear combinations of signals

Application Example: Active Noise Cancellation

A “noise” signal, that we want to get rid of.

- At subject location, signal is
  \[ x[n] \]

- Microphone picks up signal
  \[ x_c[n] \]

- Subtract the two signals:
  \[ y(t) = x(t) - x_c(t) \]

Notice careful synchronization is needed!
Then a System is a **Matrix**

\[
\begin{align*}
    y &= Du, \\
    \begin{bmatrix}
    y[1] \\
    y[2] \\
    \vdots \\
    y[M]
    \end{bmatrix} &=
    \begin{bmatrix}
    D_{11} & D_{12} & \cdots & D_{1N} \\
    D_{21} & D_{22} & \cdots & D_{2N} \\
    \vdots & \vdots & \ddots & \vdots \\
    D_{M1} & D_{M2} & \cdots & D_{MN}
    \end{bmatrix}
    \begin{bmatrix}
    u[1] \\
    u[2] \\
    \vdots \\
    u[N]
    \end{bmatrix},
\end{align*}
\]

\[
y[i] = \sum_j D_{ij}u[j].
\]
Linear Time Invariant

- Linear & Time-invariant (of course - tautology!)
- Impulse response: \( h(t) = F(\delta(t)) \)
- Why?
  - Since it is linear the output response \( (y) \) to any input \( (x) \) is:
    \[
    x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau
    \]
    \[
    y(t) = F \left[ \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau \right] = \int_{-\infty}^{\infty} x(\tau) F(\delta(t-\tau)) \, d\tau
    \]
    \[
    h(t-\tau) \equiv F(\delta(t-\tau))
    \]
    \[
    \Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau = x(t) \ast h(t)
    \]

- The output of any continuous-time LTI system is the convolution of input \( u(t) \) with the impulse response \( F(\delta(t)) \) of the system.

Linear Dynamic [Differential] System

\( \equiv \) LTI systems for which the input & output are linear ODEs

\[
a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}
\]

\[
Laplace:
\]

\[
a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)
\]

\[
A(s) Y(s) = B(s) X(s)
\]

- Total response = Zero-input response + Zero-state response
Linear Systems and ODE’s

- Linear system described by differential equation

\[ a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m} \]

- Which using Laplace Transforms can be written as

\[ a_0 Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s) \]

\[ A(s)Y(s) = B(s)X(s) \]

where \( A(s) \) and \( B(s) \) are polynomials in \( s \)

Unit Impulse Response

- \( \delta(t) \): Impulsive excitation
- \( h(t) \): characteristic mode terms

Ex:

**EXAMPLE 2.4**

Determine the unit impulse response \( h(t) \) for a system specified by the equation

\[ (s^2 + 3s + 2) y(t) = f(t) \]  

This is a second-order system \((n = 2)\) having the characteristic polynomial

\[ s^2 + 3s + 2 = (s + 1)(s + 2) \]

The characteristic roots of this system are \( s = -1 \) and \( s = -2 \). Therefore

\[ s(t) = e^{-t} + 2e^{-2t} \]  

(2.26a)

Differentiation of this equation yields

\[ s(t) = -e^{-t} - 2e^{-2t} \]  

(2.26b)

The initial conditions are free \( y(0) = 0 \) for \( n = 2 \)

\[ y(0) = 1 \quad \text{and} \quad y(0) = 0 \]

Solving \( s(t) = e^{-t} + 2e^{-2t} \) to fulfillment of initial conditions yields

\[ c_1 = 1 \quad \text{and} \quad c_2 = -1 \]

Therefore

\[ y(t) = e^{-t} - 2e^{-2t} \]

Moreover, according to Eq. 2.22, the \( s(t) \) so that

\[ s(t) = s(t) = y(t) = e^{-t} + 2e^{-2t} \]

Also if in this case, \( y(t) \) the second-order term is absent in \( s(t) \). Therefore

\[ y(t) = (F(y(t)) = e^{-t} + 2e^{-2t} \]
Where are we going with this?

This can help simplify matters…

An Example

Consider the following system:

- How to model and predict (and control the output)?

Source: EE263 (s.1-13)
This can help simplify matters…

An Example

Consider the following system:

\[ x(t) \in \mathbb{R}^8, \quad y(t) \in \mathbb{R}^1 \] → 8-state, single-output system

• Autonomous: No input yet! (\( u(t) = 0 \))
This can help simplify matters…

An Example

- Consider the following system:
Example: Let’s consider the control…

Expand the system to have a control input…

• $B \in \mathbb{R}^{8 \times 2}$, $C \in \mathbb{R}^{2 \times 8}$ (note: the 2nd dimension of C)

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = 0$$

• Problem: Find $u$ such that $y_{des}(t) = (1, -2)$

• A simple (and rational) approach:
  – solve the above equation!
  – Assume: static conditions ($u, x, y$ constant)

  ➔ Solve for $u$:

$$u_{static} = (-CA^{-1}B)^{-1}y_{des} = \begin{bmatrix} -0.63 \\ 0.36 \end{bmatrix}$$

Example: Apply $u = u_{static}$ and presto!

Note: It takes 1500 seconds for the y(t) to converge …
but that’s natural … can we do better?

Source: EE263 (s.1-13)
Example: Yes we can!

- How about:

![Graph](image1.png)

Example: **How?** How about a more clever input?

- How about:

![Graph](image2.png)

- Converges in 50 seconds (3.3% of the time 😊)

Source: EE263 (s.1-13)
Example: Can we beat it? Larger inputs & LDS

- Converges in 20 seconds (1.3% of the time 😊)

Next Time…

- We’ll talk about Other System Properties 😊

- We will introduce this via the lens of:
  “Systems as Maps. Signals as Vectors”

- Review:
  - Phasers, complex numbers, polar to rectangular, and general functional forms.
  - Chapter B and Chapter 1 of Lathi
    (particularly the first sections on signals & classification thereof)

- Register on Platypus

- Try the practise assignment