School of Information Technology and Electrical Engineering
Quiz

METR4202: ROBOTICS & AUTOMATION

August 27, 2018

First Name: ____________________________ Last Name: _____________________________________

Student Number: ________________________________________________

Examination Duration: 45 minutes
Reading Time: 10 minutes

Exam Conditions:
This is a Closed Book Quiz

Electronic Materials Permitted In The Exam Venue:
• Calculator – Allowed
• One A4 (Single-Sided) Formula Sheet – Allowed

Instructions To Students:
• Please be sure to place your name and number on ALL pages
• Please answer ALL questions.
• ⇨ ALL Answers MUST Be Justified ⇩
  (answers alone are not sufficient)

Thank you! 😊

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Total _________
1. **Putting Transformations Matrices into Perspective** [5 Points]

To get started, let’s consider the humble process of transforming frames in 3D-space using a homogeneous transformation matrix.

A transformation matrix naturally partitions into four (4) submatrices. Assuming **Rigid Body, Euclidian Space**, please specify the main **sub-components and their matrix dimensions**? (i.e., for the Transformation Matrix for a Euclidian Group)

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2. **Trigonometry Comes From Linear Algebra** [15 Points]

To get started, let’s consider two rotation matrices, \( [A] \) and \( [B] \), that are about the same origin and both rotate a vector about a 3D frame’s \( z \)-axis by the same angle \( \theta \).

(a) What are \( [A] \) and \( [B] \)?

(i.e., please expand the matrices, and give their contents/elements inside)?

(b) It intuits that the composite rotation should be \( 2\theta \).

*Please show* that \( [A][B] \) gives a rotation of \( 2\theta \).
3. Easy as $01\ldots 12\ldots 23$! [40 Points]

Consider a planar 3R robot arm with lengths $l_1$, $l_2$ and $l_3$ and frames $\{0\}$, $\{1\}$, $\{2\}$ and $\{3\}$ at the joints as shown below in Figure 1. The angles between the frames are given by $\theta_1$, $\theta_2$ and $\theta_3$ [i.e. with $\theta_1 = ^0\theta_1$, $\theta_2 = ^1\theta_2$, $\theta_3 = ^2\theta_3$].

![Figure 1: A 3R manipulator for cleaning a mirror](image)

It has the following transformation matrices between frames:

$$
^0T_1 = \begin{bmatrix}
c\theta_1 & -s\theta_1 & 0 & l_1c\theta_1 \\
s\theta_1 & c\theta_1 & 0 & l_1s\theta_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
^1T_2 = \begin{bmatrix}
c\theta_2 & -s\theta_2 & 0 & l_2c\theta_2 \\
s\theta_2 & c\theta_2 & 0 & l_2s\theta_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
^2T_3 = \begin{bmatrix}
c\theta_3 & -s\theta_3 & 0 & l_3c\theta_3 \\
s\theta_3 & c\theta_3 & 0 & l_3s\theta_3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

(a) **Spaces**

What are the **configuration space** and **workspace** for this robot arm?

(b) **Transformation**

What is the overall **Transformation Matrix** ($^0T_3$) of the tip relative to the base (at $\{0\}$)?

*Continued Overleaf…*
(c) **Looking back on this**

What is the **Transformation Matrix** \((^3T_0)\) of the base relative to the tip?

**[hint: if you are running short of time, please just outline the solution relative to \(^0T_3\)]**

(d) **Forward kinematics**

From this, what is the **Forward Kinematics** for this arm for any general three input angles \((\theta_1, \theta_2, \theta_3)\)?
Question 3 – Leaf 3

(e) Inverse kinematics
What is the Inverse Kinematics for this problem for a general end-effector pose given by $p_e = (x_e, y_e, \phi)$?

(f) More than one way to get there
How many real solutions will the Inverse Kinematics (from part (e)) have?
Will it always have this many solutions for all points $p_e$? If so, why? If not, why not?
And, in which case, what is a test for finding the number of real solutions?
(g) *A new line of reflection* [**EXTRA CREDIT**]

As noted, there is a mirror located to the right at a distance of $8\lambda$ cm away from the base (frame $\{0\}$) of the robot.

If the robot lengths are $(l_1, l_2, l_3) = (4\lambda, 2\lambda, 3\lambda)$ respectively, then please give an expression for $(\theta_1, \theta_2, \theta_3)$ such that the arm is in contact with the mirror. [i.e. if the arm were in contact with the mirror, what is the manifold of solutions?]
4. **Robots in Wonderland….** [30 Points]

Hatter, the green tea robot, is going to have a tea party! To prevent a spill (motion), for one does not want to upset the Queen (of Hearts), it needs to place the teapot on a table with exact force.

As shown in Figure 2, please assume:

- The first link, $L_1$, has mass $m_1$ and length 20 cm.
- The second link, $L_2$, has mass $m_2$ and length 15 cm.
- The teapot has mass $m_t = m_1 + m_2$ ($m_t$ is not empty) and may be considered a vertically centred 10 cm $\times$ 10 cm box (i.e. from the end of $L_2$ to the teapot’s bottom is 5 cm).
- The table is 30 cm below the base of the robot arm (i.e. $y_{table} = -30 \text{ cm}$ in frame $\{0\}$).

(a) **Forward kinematics** – Assuming the teapot angle can be controlled, what is the forward kinematics for this arm? [**Hint**: see also Question 3, part (d)]
(b) What is the Jacobian for this robot arm?
(c) If the teapot has to be placed down flat (at a pose of $0^\circ$ relative to frame $\{0\}$) with precisely \textbf{10 N} force in the vertical ($y$-axis) direction with no force in the horizontal ($x$-axis) direction.

[1] What is the reaction force on the teapot?

[2] \textit{And,} what is the resulting (additional) reaction \textbf{torques} of the contact on the robot joints?
5. **Truth in Robotics!**

Please state if the following statements are generally **TRUE (T)** or **FALSE (F)**

(Kindly circle the answer ⓞ or ⓟ, a brief justification may be **optionally** added below)

(a) **If** $A \in SE(n)$, **then** $\det A = +1$  

(b) A 3R3P manipulator must be redundant in 3D space

(c) $R_x(\frac{\pi}{2})R_y(\theta) \left( R_x(\frac{\pi}{2})^T \right) = R_z(\theta)$

(d) All elements of a Jacobian have uniform units of measure

(e) For a robot arm **with redundancies**, $\det(f) = 0$ will give the singular poses.