Position and Orientation [3]

- The rotations can be analysed based on the unit components …
- That is: the components of the orientation matrix are the unit vectors projected onto the unit directions of the reference frame

\[
\begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    (b_x) \hat{i}_B & (b_y) \hat{j}_B & (b_z) \hat{k}_B \\
    (a_x) \hat{i}_A & (a_y) \hat{j}_A & (a_z) \hat{k}_A
\end{bmatrix}
\]

Homogenous Transformation

\[
\begin{bmatrix}
    A_{RB} & A_p \\
    \gamma & \rho
\end{bmatrix}
\]

- \(\gamma\) is a projective transformation
- The Homogenous Transformation is a **linear operation** (even if projection is not)
General Coordinate Transformations [3]

- Multiple transformations compounded as a chain

\[
B_P = B_C T_C P \\
A_P = A_B T_B P \\
= A_B T_C T_C P \\
= A_C T_C P
\]

\[
A_C T = \begin{bmatrix}
A_B R_B R_C & A_P_B + A_B R_B P_C \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Tutorial Problem

The origin of frame \(\{B\}\) is translated to a position \([0, 3, 1]\) with respect to frame \(\{A\}\).

We would like to find:
1. The homogeneous transformation between the two frames in the figure.
2. For a point \(P\) defined as \([0, 1, 1]\) in frame \(\{B\}\), we would like to find the vector describing this point with respect to frame \(\{A\}\).
Tutorial Solution

- The matrix $\mathbf{T}_B^A$ is formed as defined earlier:
  
  \[
  \mathbf{T}_B^A = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 0 & -1 & 3 \\
  0 & 1 & 0 & 1 \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  \]

- Since P in the frame is:
  \[
  \mathbf{p}_B = \begin{bmatrix}
  0 \\
  1 \\
  1
  \end{bmatrix}
  \]

- We find vector $\mathbf{p}$ in frame $\{A\}$ using the relationship
  
  \[
  \mathbf{p}_A = \mathbf{T}_B^A \mathbf{p}_B
  \]

  \[
  \Rightarrow \mathbf{p}_A = \begin{bmatrix}
  0 \\
  2 \\
  2 \\
  1
  \end{bmatrix}
  \]

Example: FK/IK of a 3R Planar Arm

- Derived from Tsai (p. 63)
Example: 3R Planar Arm [2]

Position Analysis: 3·Planar 1-R Arm rotating about Z \([\mathbb{Z}]\)

\[
^0A_3 = ^0A_1 \cdot ^1A_2 \cdot ^2A_3
\]

Substituting gives:

\[
^0A_3 = \begin{bmatrix}
    C\theta_{123} & -S\theta_{123} & 0 \\
    S\theta_{123} & C\theta_{123} & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123} \\
    a_1S\theta_1 + a_2S\theta_{12} + a_3S\theta_{123} \\
    0
\end{bmatrix}
\]

Example: 3R Planar Arm [2]

Forward Kinematics

(solve for \(x\) given \(\theta \rightarrow x = f(\theta)\))

Fairly straight forward:

\[
^0R_3 = \begin{bmatrix}
    C\theta_{123} & -S\theta_{123} & 0 \\
    S\theta_{123} & C\theta_{123} & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]

\[
^0P_3 = \begin{bmatrix}
    a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123} \\
    a_1S\theta_1 + a_2S\theta_{12} + a_3S\theta_{123} \\
    0
\end{bmatrix}
\]
Example: 3R Planar Arm [3]
Inverse Kinematics
(solve for \( \theta \) given \( x \rightarrow x = f(\theta) \))

- Start with orientation \( \phi \):
  \[ C\theta_{123} = C\phi, \ S\theta_{123} = S\phi \]
  \( \Rightarrow \theta_{123} = \theta_1 + \theta_2 + \theta_3 = \phi \)

- Get overall position \( q = [q_x \ q_y] \):
  \[ q_x - a_3 C\phi = a_1 C\theta_1 + a_2 C\theta_{12} \]
  \[ q_y - a_3 S\phi = a_1 S\theta_1 + a_2 S\theta_{12} \]

Example: 3R Planar Arm [4]
- Introduce \( p = [p_x \ p_y] \) before “wrist”
  \[ p_x = a_1 C\theta_1 + a_2 C\theta_{12}, \ p_y = a_1 S\theta_1 + a_2 S\theta_{12} \]
  \( \Rightarrow p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1 a_2 C\theta_2 \)
- Solve for \( \theta_2 \):
  \( \theta_2 = \cos^{-1} \kappa, \ \kappa = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1 a_2} \) (2 \( \mathbb{R} \) roots if \( |\kappa| < 1 \))
- Solve for \( \theta_1 \):
  \[ C\theta_1 = \frac{p_x(a_1 + a_2 C\theta_2) + p_y a_2 S\theta_2}{a_1^2 + a_2^2 + 2a_1 a_2 C\theta_2}, \ S\theta_1 = \frac{-p_x a_2 S\theta_2 + p_y (a_1 + a_2 C\theta_2)}{a_1^2 + a_2^2 + 2a_1 a_2 C\theta_2} \]
  \( \theta_1 = \text{atan2}(S\theta_1, C\theta_1) \)