3. Establish the base coordinate system such that the zo-axis is aligned with the first joint axis, the xo-axis is perpendicular to the zo-axis, and the yo-axis is determined by the right-hand rule.

4. Establish the nth hand coordinate system such that the xn-axis is perpendicular to the last joint axis. The zn-axis is usually chosen in the direction of approach of the end effector.

5. Attach a Cartesian coordinate system to the distal end of all the other links as follows:
   - The zi-axis is aligned with the (i + 1)th joint axis.
   - The xi-axis is defined along the common normal between the ith and (i + 1)th joint axes, pointing from the ith to the (i + 1)th joint axis. If the joint axes are parallel, the xi-axis can be chosen anywhere perpendicular to the two joint axes. In the case of two intersecting joint axes, the xi-axis can be defined either in the direction of the vector cross product zi-1 × zi or in the opposite direction, and the origin is located at the point of intersection.
   - The yi-axis is defined according to the right-hand rule.

6. Determine the link parameters and joint variables, ai, ci, θi, and di.

There are n + 1 coordinate systems for an n-dof manipulator. However, if additional reference coordinate systems are defined, they can be related to one of the coordinate systems above by a transformation matrix. We note that John Craig used a different convention; he attached the ith coordinate system to the proximal end of link i, which results in a different homogeneous transformation matrix.

2.3 DENAVIT-HARTENBERG HOMOGENEOUS TRANSFORMATION MATRICES

Having established a coordinate system to each link of a manipulator, a 4 × 4 transformation matrix relating two successive coordinate systems can be established. Observation of Fig. 2.2 reveals that the ith coordinate system can be thought of as being displaced from the (i - 1)th coordinate system by the following successive rotations and translations.

1. The (i - 1)th coordinate system is translated along the z(i-1)-axis a distance di. This brings the origin O(i-1) into coincidence with Hi-1. The corresponding transformation matrix is

\[ T(z, d_i) = \begin{bmatrix} 1 & 0 & 0 & d_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

2. The displaced (i - 1)th coordinate system is rotated an angle θi, which brings the corresponding transformation matrix

\[ T(θ_i) = \begin{bmatrix} cθ_i & -sθ_i & 0 & 0 \\ sθ_i & cθ_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

3. The displaced (i - 1)th coordinate system is translated a distance ai. This brings the corresponding transformation matrix

\[ T(x, a_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

4. The displaced (i - 1)th coordinate system is rotated an angle αi, which brings the corresponding transformation matrix

\[ T(x, α_i) = \begin{bmatrix} cα_i & -sα_i & 0 & 0 \\ sα_i & cα_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

We may think of the transformation matrix, \( i^{-1}A_i \), as being displaced from the (i - 1)th coordinate system by the following successive rotations and translations.

\[ i^{-1}A_i = T(z, d_i) \cdot T(θ_i) \cdot T(x, a_i) \cdot T(x, α_i) \]

Expanding Eq. (2.1), we obtain

\[ i^{-1}A_i = \begin{bmatrix} cθ_i & -sθ_i & 0 & a_i \\ sθ_i & cθ_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
ch that the $z_0$-axis is aligned perpendicular to the $z_0$-axis, and d rule.

1 such that the $x_n$-axis is perpendicular to the $z_0$-axis is usually chosen in the the distal end of all the other

)th joint axis.

mon normal between the $i$th and the $(i+1)$th joint $x_i$-axis can be chosen any-axises. In the case of two inter-defined either in the direction of intersection.

ie right-hand rule.

iables, $a_i$, $\alpha_i$, $\theta_i$, and $d_i$.

n-dof manipulator. However, defined, they can be related to the moving coordinate matrix. We note that attached the $i$th coordinate is in a different homogeneous

NEOUS

link of a manipulator, a $4 \times 2$ coordinate systems can be at the $i$th coordinate system and $1$th coordinate system by transformed along the $z_{i-1}$-axis a distance $d_i$. The

\[
T(z, d) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

2. The displaced $(i - 1)$th coordinate system is rotated about the $z_{i-1}$-axis an angle $\theta_i$, which brings the $x_{i-1}$-axis into alignment with the $x_i$-axis. The corresponding transformation matrix is

\[
T(z, \theta) = \begin{bmatrix}
c\theta_i & -s\theta_i & 0 & 0 \\
s\theta_i & c\theta_i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

3. The displaced $(i - 1)$th coordinate system is translated along the $x_i$-axis a distance $a_i$. This brings the origin $O_{i-1}$ into coincidence with $O_i$. The corresponding transformation matrix is

\[
T(x, a) = \begin{bmatrix}
1 & 0 & 0 & a_i \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

4. The displaced $(i - 1)$th coordinate system is rotated about the $x_i$-axis an angle $\alpha_i$, which brings the two coordinate systems into complete coincidence. The corresponding transformation matrix is

\[
T(x, \alpha) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & c\alpha_i & -s\alpha_i & 0 \\
0 & s\alpha_i & c\alpha_i & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

We may think of the transformations above as four basic transformations about the moving coordinate axes. Therefore, the resulting transformation matrix, $i^{-1}A_i$, is given by

\[
i^{-1}A_i = T(z, d)T(z, \theta)T(x, a)T(x, \alpha).
\] \hspace{1cm} (2.1)

Expanding Eq. (2.1), we obtain

\[
i^{-1}A_i = \begin{bmatrix}
c\theta_i & -s\theta_i & s\alpha_i & c\alpha_i & a_i c\theta_i \\
s\theta_i & c\theta_i & s\alpha_i & c\alpha_i & a_i s\theta_i \\
0 & s\alpha_i & c\alpha_i & \alpha_i & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}.
\] \hspace{1cm} (2.2)
Equation (2.2) is called the Denavit–Hartenberg (D-H) transformation matrix. The trailing subscript $i$ and the leading superscript $i - 1$ denote that the transformation takes place from the $i$th coordinate system to the $(i - 1)$th coordinate system.

Let the homogeneous coordinates of the position vector of a point relative to the $i$th coordinate system be denoted by $\mathbf{x} = [p_x, p_y, p_z, 1]^T$. Also let the homogeneous coordinates of a unit vector expressed in the $i$th coordinate system be denoted by $\mathbf{u} = [u_x, u_y, u_z, 0]^T$. Then the transformation of a position vector and a unit vector from the $i$th to the $(i - 1)$th coordinate system can be written as

\[
\begin{align*}
\mathbf{x}_{i-1} &= i^{-1}A_i i^1\mathbf{x}, \\
\mathbf{u}_{i-1} &= i^{-1}A_i i^1\mathbf{u}.
\end{align*}
\]

Note that the leading superscript is used to indicate the coordinate system with respect to which a vector is expressed. Although the transformation matrix $A$ is not orthogonal, the inverse transformation exists and is given by

\[
i^{-1}A_i (i-1) = \begin{bmatrix}
c\theta_i & s\theta_i & 0 & -a_i \\
-c\alpha_i s\theta_i & c\alpha_i c\theta_i & s\alpha_i & -d_i s\alpha_i \\
s\alpha_i s\theta_i & -s\alpha_i c\theta_i & c\alpha_i & -d_i c\alpha_i \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Example 2.3.1 Planar 3-DOF Manipulator

Figure 2.3 shows a 3-dof planar manipulator constructed with three revolute joints located at points $O_0$, $A$, and $P$, respectively. A coordinate system is attached to each link. The $(x_0, y_0, z_0)$ coordinate system is attached to the base with its origin located at the first joint pivot and the $x$-axis pointing to the right. Since the joint axes are all parallel to each other, all the twist angles $\alpha_i$ and translational distances $d_i$ are zero.

For the coordinate systems chosen, the link parameters are given in Table 2.1. The D-H transformation matrices are obtained by substituting the D-H link parameters into Eq. (2.2):

\[
\begin{bmatrix}
c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\
s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

Example 2.3.2 SCARA Arm

dof manipulator. It has been pr
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(2.6)

Also let \( p = [p_x, p_y, p_z] \) denote the position vector of a point relative to the \( (i-1) \)th coordinate system. Then the transformation of a point from the \( i \)th to the \( (i-1) \)th coordinate system is given by

\[
T_i^i-1 = \begin{bmatrix}
    c_{\theta_1} & s_{\theta_1} & 0 & 0 \\
    -s_{\theta_1}c_{\alpha_1} & c_{\theta_1}c_{\alpha_1} & s_{\theta_1}c_{\alpha_1} & 0 \\
    -s_{\theta_1}s_{\alpha_1} & c_{\theta_1}s_{\alpha_1} & c_{\alpha_1} & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

(2.3)

Example 2.2.2. SCARA Arm

The SCARA arm is an important type of 4-DOF manipulator. It has been produced by several companies, including Adept.

<table>
<thead>
<tr>
<th>Joint</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

\[
T_3^2 = \begin{bmatrix}
    s_{\theta_3} & -c_{\theta_3} & 0 & 0 \\
    c_{\theta_3}s_{\alpha_3} & c_{\theta_3}c_{\alpha_3} & -s_{\theta_3} & 0 \\
    -c_{\theta_3}s_{\alpha_3} & c_{\theta_3}c_{\alpha_3} & s_{\theta_3} & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

(2.8)

\[
T_2^1 = \begin{bmatrix}
    c_{\theta_2} & s_{\theta_2} & 0 & 0 \\
    -s_{\theta_2}c_{\alpha_2} & c_{\theta_2}c_{\alpha_2} & s_{\theta_2}c_{\alpha_2} & 0 \\
    -s_{\theta_2}s_{\alpha_2} & c_{\theta_2}s_{\alpha_2} & c_{\alpha_2} & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

(2.7)

**Figure 2.3**. Planar 3-dof manipulator.
Technologically, IBM, Seiko, and others. A SCARA arm is constructed with four joint axes parallel to each other. The first two and the fourth are revolute joints, and the third is a prismatic joint. Figure 2.4 shows a schematic diagram of a SCARA arm. For the coordinate systems established in the figure, the corresponding link parameters are listed in Table 2.2.

Substituting the D-H link parameters into Eq. (2.2), we obtain the D-H transformation matrices:

\[
0_A_1 =\begin{bmatrix}
\cos\theta_1 & -\sin\theta_1 & 0 & a_1\cos\theta_1 \\
\sin\theta_1 & \cos\theta_1 & 0 & a_1\sin\theta_1 \\
0 & 0 & 1 & d_1 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

TABLE 2.2. D-H Parameters of the SCARA Arm

<table>
<thead>
<tr>
<th>Joint i</th>
<th>(a_i)</th>
<th>(a_i)</th>
<th>(d_i)</th>
<th>(\theta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>(a_1)</td>
<td>(d_1)</td>
<td>(\theta_1)</td>
</tr>
<tr>
<td>2</td>
<td>(\pi)</td>
<td>(a_2)</td>
<td>0</td>
<td>(\theta_2)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>(d_3)</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>(d_4)</td>
<td>(\theta_4)</td>
</tr>
</tbody>
</table>

In this robot, the joint variables control the \(x\) and \(y\) coordinates, and the fourth joint control the \(z\) coordinate, and the fourth joint is a prismatic joint. The orientation of the end effector cannot be specified and must be always pointing in the direction of the end effector. Since the robot has only 4 degrees of freedom, the end effector cannot be specified and must be always pointing in the direction of the robot.

2.4 LOOP-CLOSURE EQUATIONS

In a study of the kinematics of robotic systems, an algebraic equation relating the variables is sought. The location of the end effector is specified by a point \(Q\), and the orientation is given by the three Euler angles, \(u\), \(v\), and \(w\). If the \(w-u\) rotation matrix is

\[
\begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

where the upper right \(3 \times 1\) submatrix is the end effector point \(Q\) and the upper left \(3 \times 3\) submatrix is the orientation of the end effector. The orientation of the end effector cannot be specified and must be always pointing in the direction of the robot.
A joint 3 Z4 SCARA arm is constructed with four o and the fourth are revolute ~.4 shows a schematic diagram i established in the figure, the IIe 2.2.

In this robot, the joint variables are \( \theta_1, \theta_2, d_3, \) and \( \theta_4 \). The first two joint variables control the x and y coordinates, the third joint variable controls the z coordinate, and the fourth joint variable controls the orientation of the end effector. Since the robot has only 4 degrees of freedom, the orientation of the end effector cannot be specified arbitrarily. As a matter of fact, the \( z_4 \)-axis must be always pointing in the negative \( z_0 \) direction. Although the SCARA robot has only 4 degrees of freedom, it is very useful for assembling components on a plane such as a PC board.

### 2.4 LOOP-CLOSURE EQUATIONS

In a study of the kinematics of robot manipulators, we are interested in deriving an algebraic equation relating the location of the end effector to the joint variables. The location of the end effector can be specified by the following 4 x 4 homogeneous transformation matrix:

\[
^{0}A_n = \begin{bmatrix} u & v & w & q \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

where the upper right 3 x 1 submatrix describes the position of a reference point \( Q \) and the upper left 3 x 3 submatrix describes the orientation of the end effector. The orientation of the end effector can be specified in terms of three Euler angles, or the direction cosines of the three end-effector coordinate axes, \( u, v, \) and \( w \). If the \( w-u-w \) Euler angles are used, for example, the elements of the upper left 3 x 3 submatrix are given by

\[
\begin{align*}
  u_x &= c\phi c\psi - s\phi c\theta s\psi, \\
  u_y &= s\phi c\psi + c\phi c\theta s\psi,
\end{align*}
\]
\[ u_x = s\theta s\phi, \]
\[ u_y = -c\phi s\psi - s\phi c\theta c\psi, \]
\[ u_z = s\phi c\phi c\psi, \]
\[ v_x = -c\phi s\psi - s\phi c\theta c\psi, \]
\[ v_y = -s\phi s\psi + c\phi c\theta c\psi, \]
\[ v_z = s\phi c\psi, \]
\[ w_x = s\phi s\phi, \]
\[ w_y = -c\phi s\phi, \]
\[ w_z = c\phi. \]

If the direction cosines are used, \( u, v, \) and \( w \) represent three unit vectors directed along the three coordinate axes of the hand coordinate system and expressed in the base coordinate system.

From the geometry of the links, the transformation matrix \( {}^0A_n \) above can be thought of as the resultant of a series of coordinate transformations beginning from the base coordinate system to the end-effector coordinate system. That is,

\[ {}^0A_1^{-1}A_2^{-1}A_3^{-1} \ldots {}^0A_n^{-1} = {}^0A_n. \]  

Equation (2.15) is called the loop-closure equation of a serial manipulator. It contains 16 scalar equations, four of which are trivial. Equating the upper right \( 3 \times 1 \) submatrix results in three independent equations, representing the position of the end effector. Equating the elements of the upper left \( 3 \times 3 \) submatrix results in nine equations, representing the orientation of the end effector. However, only three of the nine orientation equations are independent because of the orthogonal conditions.

The loop-closure equation, Eq. (2.15), can be used to solve both direct and inverse kinematics problems. For direct kinematics, the joint variables are given and the problem is to find where the end effector is with respect to the base coordinate system. This can be accomplished by multiplying the D-H matrices on the left-hand side of the equation. For the inverse kinematics, the end-effector location (i.e., \( {}^0A_n \)) is given and the problem is to find the joint variables needed to bring the end effector to the desired location. The problem becomes very nonlinear. In what follows, we concentrate on the inverse kinematics problem.

**Example 2.4.1 Scorbot Robot** Figure 2.5 shows a schematic diagram of the Scorbot robot. In this diagram, the second, third, and fourth joint axes are parallel to one another and point into the paper at points \( A, B, \) and \( P, \) respectively. The first joint axis points up vertically, and the fifth joint axis intersects the fourth perpendicularly. We wish to find the overall transformation matrix for the robot.
represent three unit vectors in the hand coordinate system and the orientation matrix $\mathbf{A}_n$ above can be written as the product of separate transformations beginning with the end-effector coordinate system. That is,

$$\mathbf{A}_n = \mathbf{R}_n \mathbf{T}_n \mathbf{R}_n^{-1} \mathbf{T}_n^{-1}.$$
\[
\begin{align*}
\mathbf{1}_A &= \begin{bmatrix}
  c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\
  s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix},
\tag{2.17}
\\
\mathbf{2}_A &= \begin{bmatrix}
  c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 \\
  s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix},
\tag{2.18}
\\
\mathbf{3}_A &= \begin{bmatrix}
  c\theta_4 & -s\theta_4 & 0 & a_4 c\theta_4 \\
  s\theta_4 & c\theta_4 & 0 & a_4 s\theta_4 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix},
\tag{2.19}
\\
\mathbf{4}_A &= \begin{bmatrix}
  c\theta_5 & -s\theta_5 & 0 & 0 \\
  s\theta_5 & c\theta_5 & 0 & 0 \\
  0 & 0 & 1 & d_5 \\
  0 & 0 & 0 & 1
\end{bmatrix}.
\tag{2.20}
\end{align*}
\]

Multiplying Eqs. (2.17), (2.18), and (2.19) yields
\[
\mathbf{1}_A = \begin{bmatrix}
  c\theta_{234} & -s\theta_{234} & a_3 c\theta_{23} + a_2 c\theta_2 \\
  s\theta_{234} & c\theta_{234} & a_3 s\theta_{23} + a_2 s\theta_2 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix},
\tag{2.21}
\]

where \(c\theta_{ij} = \cos(\theta_i + \theta_j), s\theta_{ij} = \sin(\theta_i + \theta_j), c\theta_{ijk} = \cos(\theta_i + \theta_j + \theta_k),\) and \(s\theta_{ijk} = \sin(\theta_i + \theta_j + \theta_k).\)

Note that Eq. (2.21) provides a transformation from the fourth coordinate system to the first coordinate system. We may treat \(\theta_2, \theta_23,\) and \(\theta_{234}\) as new variables. In this way, the orientation submatrix contains only one variable, \(\theta_{234}\), while the position submatrix contains two variables, \(\theta_2\) and \(\theta_{23}\). This important fact can be used for deriving a closed-form solution for any manipulator with three consecutive parallel joint axes.

Multiplying Eqs. (2.16), (2.21), and (2.20) yields the elements of the overall transformation matrix \(A_1:\)
\[
\begin{align*}
u_x &= c\theta_1 c\theta_{234} c\theta_3 + s\theta_1 s\theta_3, \\
u_y &= s\theta_1 c\theta_{234} c\theta_5 - c\theta_1 s\theta_5, \\
u_z &= -s\theta_{234} c\theta_5, \\
u_x &= -c\theta_1 c\theta_{234} s\theta_5 + s\theta_1 c\theta_5,
\end{align*}
\]

Since this is a 5-dof manipulator, the overall transformation matrix \(A_0\) may be specified. Values of the direction of a line in the cartesian coordinate system can be specified by the position of the point of interest and direction of a line are easy.

### 2.5 OTHER COORDINATE SYSTEMS

In the preceding section, the Cartesian coordinate system was chosen to be in line with the first coordinate system. Another coordinate system, defined in terms of the tool frame, is defined.

\[
\begin{align*}
\mathbf{A}_0 &= \begin{bmatrix}
  c\theta_1 & c\theta_1 & c\theta_1 & c\theta_1 & c\theta_1 \\
  s\theta_1 & s\theta_1 & s\theta_1 & s\theta_1 & s\theta_1 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

where \(\mathbf{A}_0\) and \(A_0\) are the same.

### 2.6 DENAVIT–HARTFORD (DH)

Although the loop-closure operation is in the inverse kinematics problem. In general, if there are three or more axes, we may combine the
\[
\begin{bmatrix}
2c\theta_2 \\
2s\theta_2 \\
0 \\
1
\end{bmatrix}, \quad (2.17)
\]
\[
\begin{bmatrix}
1c\theta_3 \\
1s\theta_3 \\
0 \\
1
\end{bmatrix}, \quad (2.18)
\]
\[
\begin{bmatrix}
1c\theta_4 \\
1s\theta_4 \\
0 \\
1
\end{bmatrix}, \quad (2.19)
\]
\[
\begin{bmatrix}
1c\theta_5 \\
s\theta_5 \\
0 \\
1
\end{bmatrix}, \quad (2.20)
\]
\[
\begin{bmatrix}
\lambda_0 + a_2c\theta_2 \\
\lambda_0 + a_2s\theta_2 \\
0 \\
1
\end{bmatrix}, \quad (2.21)
\]
\[
\gamma_{jk} = \cos(\theta_j + \theta_k + \theta_k), \quad (2.22)
\]

Since this is a 5-dof manipulator, only five of the six parameters of the end effector can be specified. Very often, the desired position of a point and the direction of a line in the end effector (e.g., the position of point Q and the direction of \(x_5\)-axis) are specified. Five-dof manipulators are useful for spray painting, spot welding, and sealant applications for which only the position and direction of a line are essential.

### 2.5 OTHER COORDINATE SYSTEMS

In the preceding section, the \(z_0\)-axis of the base coordinate system was chosen to be in line with the first joint axis, and the \(z_n\)-axis of the hand coordinate system was chosen to be in the direction of approach. If an additional coordinate system is defined in the base with a transformation matrix \(^{ref}A_0\), and another coordinate system is defined in the tool frame with a transformation matrix \(^{\text{tool}}A_0\), the overall loop-closure equation can be modified as

\[
^{\text{tool}}A_0 = ^{ref}A_0 \cdot ^{\text{tool}}A_0 \cdot ^{\text{tool}}A_0, \quad (2.23)
\]

where \(^{ref}A_0\) and \(^{\text{tool}}A_0\) are constant transformation matrices.

### 2.6 DENAVIT–HARTENBERG METHOD

Although the loop-closure equation, Eq. (2.15), can be applied to solve the inverse kinematics problem, in practice it is rarely solved in its present form. In general, if there are three intersecting joint axes, we may work with the position of the point of intersection first, thereby avoiding the joint variables associated with the three intersecting axes. If there are three parallel joint axes, we may combine the three joint variables as illustrated in the Scorbot.
robot example. We may also pre- or postmultiply the loop-closure equation by the inverse of the matrix \( ^{i-1}A_i \) to obtain alternative loop-closure equations, such as

\[
(^{0}A_1)^{-1} ^{0}A_n = ^{1}A_2 ^{2}A_3 \ldots ^{n-1}A_n, \quad (2.24)
\]

\[
(^{1}A_2)^{-1}(^{0}A_1)^{-1} ^{0}A_n = ^{2}A_3 ^{3}A_4 \ldots ^{n-1}A_n, \quad (2.25)
\]

\[
(^{2}A_3)^{-1}(^{1}A_2)^{-1}(^{0}A_1)^{-1} ^{0}A_n = ^{3}A_4 ^{4}A_5 \ldots ^{n-1}A_n. \quad (2.26)
\]

One reason for rearranging the loop-closure equation is to redistribute the unknown variables on both sides of the equation as evenly as possible. Another reason is to take advantage of some special conditions, such as three consecutive intersecting joint axes or three consecutive parallel joint axes. In many cases, the equation becomes decoupled and a closed-form solution can be derived.

### 2.6.1 Position Analysis of a Planar 3-DOF Manipulator

For the planar 3-dof manipulator shown in Fig. 2.3, the overall transformation matrix is given by

\[
^{0}A_3 = ^{0}A_1 ^{1}A_2 ^{2}A_3. \quad (2.27)
\]

Substituting Eqs. (2.6) through (2.8) into (2.27), we obtain

\[
^{0}A_3 = \begin{bmatrix}
  c_{\theta_{123}} & -s_{\theta_{123}} & 0 & a_1 c_{\theta_1} + a_2 c_{\theta_{12}} + a_3 c_{\theta_{123}} \\
  s_{\theta_{123}} & c_{\theta_{123}} & 0 & a_1 s_{\theta_1} + a_2 s_{\theta_{12}} + a_3 s_{\theta_{123}} \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}. \quad (2.28)
\]

(a) Direct Kinematics. The position vector of the origin \( Q \) expressed in the end-effector coordinate system is given by \( ^{3}q = [0, 0, 0, 1]^T \). Let the position vector of \( Q \) with respect to the base coordinate system be \( ^{0}q = [q_x, q_y, q_z]^T \). Then we can relate \( ^{3}q \) to \( ^{0}q \) by the following transformation:

\[
\begin{bmatrix}
  q_x \\
  q_y \\
  q_z
\end{bmatrix} = ^{0}A_3 \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix} = \begin{bmatrix}
  a_1 c_{\theta_1} + a_2 c_{\theta_{12}} + a_3 c_{\theta_{123}} \\
  a_1 s_{\theta_1} + a_2 s_{\theta_{12}} + a_3 s_{\theta_{123}} \\
  0 \\
  1
\end{bmatrix}. \quad (2.29)
\]

Hence, given \( \theta_1, \theta_2, \) and \( \theta_3 \), the position of point \( Q \) can be computed by Eq. (2.29). Similarly, the position vector of any other point in the end effector, \( ^{3}g = [g_x, g_y, 0, 1]^T \), is given by

\[
\begin{bmatrix}
  g_x \\
  g_y \\
  g_z
\end{bmatrix} = ^{0}A_3 \begin{bmatrix}
  g_{u} \\
  g_{v} \\
  1
\end{bmatrix} = \begin{bmatrix}
  g_{u} c_{\theta_{12}} \\
  g_{u} s_{\theta_{12}} \\
  g_{v}
\end{bmatrix}.
\]

From Eq. (2.28), we conclude that \( \theta_1 + \theta_2 + \theta_3 \) equal to \( \theta_1 + \theta_2 + \theta_3 \).

(b) Inverse Kinematics. For the planar 3-dof manipulator, the end effect of point \( Q \) and an orientation any transformation matrix from the \( A_3 \) coordinate system, \( ^{0}A_3 \), is given by

\[
^{0}A_3 = \begin{bmatrix}
  c_{\theta_{123}} & -s_{\theta_{123}} & 0 & a_1 c_{\theta_1} + a_2 c_{\theta_{12}} + a_3 c_{\theta_{123}} \\
  s_{\theta_{123}} & c_{\theta_{123}} & 0 & a_1 s_{\theta_1} + a_2 s_{\theta_{12}} + a_3 s_{\theta_{123}} \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}.
\]

Inverse kinematics solutions can equate the (1,1) and (2,1) elements of Eq. (2.28) to that of (2.31). To equate the (1,1) and (2,1) elements,

\[
\begin{align*}
\theta_{123} &= \text{solve}\, c_{\theta_{123}} = \text{constant}\, \text{for}\, \theta_{123} \\
\end{align*}
\]

Hence

\[
\theta_{123} = \text{solve}\, c_{\theta_{123}} = \text{constant}\, \text{for}\, \theta_{123} \\
\]

Next, we equate the (1,4) and (2,4) elements,

\[
\begin{align*}
p_x &= q_x - a_3 c_{\theta_3} \\
p_y &= q_y - a_3 s_{\theta_3} \\
\end{align*}
\]

where \( p_x = q_x - a_3 c_{\theta} \) and \( p_y = q_y - a_3 s_{\theta} \), point \( P \) located at the third joint.
iply the loop-closure equation
ative loop-closure equations,
\[ 2A_1 \cdots a^{-1}A_n, \quad (2.24) \]
\[ 3A_4 \cdots a^{-1}A_n, \quad (2.25) \]
\[ 4A_5 \cdots a^{-1}A_n. \quad (2.26) \]

equation is to redistribute the
fraction as evenly as possible. An-

cial conditions, such as three
secutive parallel joint axes. In
and a closed-form solution can

OF Manipulator

2.3, the overall transformation

\[ \begin{bmatrix}
    \theta_1 + \theta_2 + \theta_3 \\
    \phi
\end{bmatrix} = \begin{bmatrix}
    a_1c_1 + a_2c_2 + a_3c_3 \\
    a_1s_1 + a_2s_2 + a_3s_3
\end{bmatrix} \quad (2.29)
\]

pt Q can be computed by
other point in the end effector,

\[ g = [g_x, g_y, 0, 1]^T, \] is given by
\[ \begin{bmatrix}
    g_x \\
    g_y \\
    g_z
\end{bmatrix} = \begin{bmatrix}
    a_1c_1 + a_2c_2 + a_3c_3 \\
    a_1s_1 + a_2s_2 + a_3s_3
\end{bmatrix} \quad (2.30) \]

From Eq. (2.28), we conclude that the orientation angle of the end effector is equal to \( \theta_1 + \theta_2 + \theta_3 \).

(b) Inverse Kinematics. For the inverse kinematics problem, the location of the end effector is given and the problem is to find the joint angles \( \theta_i, i = 1, 2, 3 \), necessary to bring the end effector to the desired location. For a planar 3-dof manipulator, the end effector can be specified in terms of the position of point Q and an orientation angle \( \phi \) of the end effector. Hence the overall transformation matrix from the end-effector coordinate system to the base coordinate system, \( ^0A_3 \), is given by

\[ ^0A_3 = \begin{bmatrix}
    c\phi & -s\phi & 0 & a_x \\
    s\phi & c\phi & 0 & a_y \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \quad (2.31) \]

Inverse kinematics solutions can be obtained by equating the elements of Eq. (2.28) to that of (2.31). To find the orientation of the end effector, we equate the (1,1) and (2,1) elements of Eq. (2.28) to that of (2.31):
\[ c\theta_{123} = c\phi, \quad (2.32) \]
\[ s\theta_{123} = s\phi. \quad (2.33) \]

Hence
\[ \theta_{123} = \theta_1 + \theta_2 + \theta_3 = \phi. \quad (2.34) \]

Next, we equate the (1,4) and (2,4) elements of Eq. (2.28) to that of (2.31):
\[ p_x = a_1c_1 + a_2c_2, \quad (2.35) \]
\[ p_y = a_1s_1 + a_2s_2. \quad (2.36) \]

where \( p_x = q_x - a_3c\phi \) and \( p_y = q_y - a_3s\phi \) denote the position vector of the point P located at the third joint axis shown in Fig. 2.3. Note that by using
this substitution, \( \theta_3 \) disappears from Eqs. (2.35) and (2.36). From Fig. 2.3 we observe that the distance from point \( O \) to \( P \) is independent of \( \theta_1 \). Hence we can eliminate \( \theta_1 \) by summing the squares of Eqs. (2.35) and (2.36); that is,

\[
p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1a_2\cos \theta_2.
\]  

(2.37)

Solving Eq. (2.37) for \( \theta_2 \), we obtain

\[
\theta_2 = \cos^{-1} \kappa,
\]  

(2.38)

where

\[
\kappa = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1a_2}.
\]

Equation (2.38) yields (1) two real roots if \( |\kappa| < 1 \), (2) one double root if \( |\kappa| = 1 \), and (3) no real roots if \( |\kappa| > 1 \). In general, if \( \theta_2 = \theta_2^* \) is a solution, \( \theta_2 = -\theta_2^* \) is also a solution, where \( \pi \geq \theta_2^* \geq 0 \). We call \( \theta_2 = \theta_2^* \) the elbow-down solution and \( \theta_2 = -\theta_2^* \) the elbow-up solution. If \( |\kappa| = 1 \), the arm is in a fully stretched or folded configuration. If \( |\kappa| > 1 \), the position is not reachable.

Corresponding to each \( \theta_2 \), we can solve \( \theta_1 \) by expanding Eqs. (2.35) and (2.36) as follows:

\[
(a_1 + a_2\cos \theta_2)c\theta_1 - (a_2\sin \theta_2)s\theta_1 = p_x,
\]  

(2.39)

\[
(a_2\sin \theta_2)c\theta_1 + (a_1 + a_2\cos \theta_2)s\theta_1 = p_y,
\]  

(2.40)

Solving Eqs. (2.39) and (2.40) for \( c\theta_1 \) and \( s\theta_1 \), yields

\[
c\theta_1 = \frac{p_x(a_1 + a_2\cos \theta_2) + p_ya_2\sin \theta_2}{\Delta},
\]

\[
s\theta_1 = \frac{-p_xa_2\sin \theta_2 + p_y(a_1 + a_2\cos \theta_2)}{\Delta},
\]

where \( \Delta = a_1^2 + a_2^2 + 2a_1a_2\cos \theta_2 \). Hence, corresponding to each \( \theta_2 \), we obtain a unique solution for \( \theta_1 \):

\[
\theta_1 = \text{Atan2}(s\theta_1, c\theta_1).
\]  

(2.41)

In a computer program we may use the function \( \text{Atan2}(x, y) \) to obtain a unique solution for \( \theta_1 \). However, the solution may be real or complex. A complex solution corresponds to an end-effector location that is not reachable by the manipulator. Once \( \theta_1 \) and \( \theta_2 \) are known, Eq. (2.34) yields a unique solution for \( \theta_3 \). Hence, corresponding to a given end-effector location, there are generally two real inverse kinematics solutions, one of them lying in the plane defined by the other about a line connecting the origin \( O \) and \( \theta_2 \).

Vector-Loop Method. Although a powerful tool, the inverse kinematic methods, such as the vector-loop method, becomes more efficient for analysis purposes as the manipulator's workspace increases. For convenience, the reader is referred to Figure 2.3. For convenience, the reader is referred to Figure 2.7.

Using vector algebra, the position of the end-effector relative to the origin \( Q \) of the end-effector is given by

\[
\text{vector-loop method: } \Delta.
\]

From Fig. 2.7, we observe that the end-effector angles by

\[
\phi.
\]

We now formulate the vector-loop method.
and (2.36). From Fig. 2.3 we independent of $\theta_1$. Hence we s. (2.35) and (2.36); that is,

$$I \propto \theta_2 c \theta_2.$$  
(2.37)

$$I \propto \theta_2^2.$$  
(2.38)

$\theta = 1$, (2) one double root if
general, if $\theta_2 = \theta_2^*$ is a solution,
$\phi$, we call $\theta_2 = \theta_2^*$ the elbow-solution. If $|\phi| = 1$, the arm is
$|\phi| > 1$, the position is not

$\phi$ expanding Eqs. (2.35) and

$$h_1 = p_x,$$  
(2.39)

$$h_1 = p_y.$$  
(2.40)

yields

$$a_2 s \theta_2,$$  
(2.41)

$$a_2 c \theta_2.$$  
(2.42)

onding to each $\theta_2$, we obtain

$$a_3 c \theta_1,$$  
(2.43)

$\theta_1$ to obtain a

$\theta_2$ yields a unique solu-
l-end-effector location, there are
generally two real inverse kinematics solutions, one being the reflection of
the other about a line connecting points $O$ and $P$, as illustrated in Fig. 2.6.

**Vector-Loop Method.** Although the D-H method of analysis is a very pow-
erful tool, the inverse kinematics problem can often be solved by other

methods, such as the vector-loop method. For example, the vector-loop method
becomes more efficient for analysis of the 3-dof planar manipulator shown in
Fig. 2.3. For convenience, the manipulator has been resketched as shown in
Fig. 2.7.

Using vector algebra, the position vector of the wrist center $P$ can be re-
lated to the origin $Q$ of the end effector by the equations

$$p_x = q_x - a_3 c \phi,$$  
(2.42)

$$p_y = q_y - a_3 s \phi.$$  
(2.43)

From Fig. 2.7, we observe that the orientation angle $\phi$ is related to the joint
angles by

$$\phi = \theta_1 + \theta_2 + \theta_3.$$  
(2.44)

We now form a fictitious vector loop equation as follows:

$$OA + AP = OP.$$  
(2.45)
Taking the x and y components of Eq. (2.45) yields

\[ p_x = a_1 c \theta_1 + a_2 c \theta_1 \]  
\[ p_y = a_1 s \theta_1 + a_2 s \theta_1 \]  

Note that using the vector-loop method, we have derived Eqs. (2.35) and (2.36) with very little effort.

### 2.6.2 Position Analysis of the Scorbot Robot

For the Scorbot robot shown in Fig. 2.5, the overall transformation matrix is given by Eq. (2.22). We wish to solve the direct and inverse kinematics problems.

**(a) Direct Kinematics.** For the direct kinematics problem, we simply substitute the given joint angles into Eq. (2.22) to obtain the end-effector position, \((q_x, q_y, q_z)\), and the orientation in terms of the three unit vectors \((u_x, u_y, u_z)\), \((v_x, v_y, v_z)\), and \((w_x, w_y, w_z)\).

**(b) Inverse Kinematics.** For the inverse kinematics problem, only 5 of the 12 parameters associated with the end-effector position vector and rotation matrix can be specified at will. This is because the manipulator has only 5 degrees of freedom. It is obvious that the position vector \(q\) and the approach vector \(w\) cannot be specified together depend only on 4 dofs. In the exercise we assume that \(q\) and \(w\) are not known and are to be determined.

Although Eq. (2.22) can be rewritten in the following form, we take a more straightforward approach to solve the loop-closure equation:

\[ (0_{1}^{1}) \]

Equating the first column:

\[ u_x c \theta_1 + u_y c \theta_1 + u_z s \theta_1 = q_x \]

Similarly, equating the fourth column:

\[ v_x c \theta_1 + v_y c \theta_1 + v_z s \theta_1 = q_z \]

The first joint angle, \(\theta_1\), is found by solving the equation:

\[ c \theta_1 = \frac{q_x}{u_x} \]

There are two solutions; that is, \(\theta_1 = \theta^{*}_1\) or \(\theta_1 = \theta^{*}_1\). Once \(\theta_1\) is found, the second joint angle, \(\theta_2\), is found by solving:

\[ \theta_2 = \text{Atan2}(v_y, v_x) \]

That is, if \(\theta_2 = \theta^{*}_2\) is a solution. Corresponding to each solution of \(\theta_2\) we produce a unique solution of \(\theta_3\):

\[ \theta_{34} = \text{Atan2}(w_y, w_x) \]

Next, we solve Eqs. (2.52) (2.53) can be written
vector \( w \) cannot be specified simultaneously, due to the fact that \( q \) and \( w \) together depend only on 4 degrees of freedom of the manipulator. For this exercise we assume that \( q \) and \( u \) are specified and that the other two unit vectors, \( v \) and \( w \), are to be determined after the joint angles are found.

Although Eq. (2.22) can be used to solve the inverse kinematics, in what follows we take a more straightforward approach by multiplying both sides of the loop-closure equation by \((A_1^o)^{-1}\); that is,

\[
(A_1^o)^{-1} A_5 = A_2^o A_3^o A_4^o A_5. \tag{2.48}
\]

Equating the first column of Eq. (2.48), we obtain

\[
\begin{align*}
u_x c\theta_1 + u_y s\theta_1 &= c\theta_{234} c\theta_5, \\
u_z &= s\theta_{234} c\theta_5,
\end{align*}
\tag{2.49}
\]
\[
\begin{align*}
u_x s\theta_1 + u_z c\theta_1 &= -s\theta_5. 
\end{align*}
\tag{2.50}
\]

Similarly, equating the fourth column of Eq. (2.48), we obtain

\[
\begin{align*}
q_x c\theta_1 + q_y s\theta_1 - a_1 &= a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234}, \\
-q_x + d_1 &= a_2 s\theta_2 + a_3 s\theta_{23} + d_5 c\theta_{234}, \\
-q_z s\theta_1 + q_y c\theta_1 &= 0.
\end{align*}
\tag{2.51}
\]

The first joint angle, \( \theta_1 \), is obtained immediately from Eq. (2.54):

\[
\theta_1 = \tan^{-1} \frac{q_y}{q_x}. \tag{2.55}
\]

There are two solutions; that is, if \( \theta_1 = \theta_1^* \) is a solution, \( \theta_1 = \pi + \theta_1^* \) is also a solution. Once \( \theta_1 \) is found, two solutions for \( \theta_2 \) are obtained from Eq. (2.51):

\[
\theta_2 = \sin^{-1}(u_z s\theta_1 - u_x c\theta_1). \tag{2.56}
\]

That is, if \( \theta_2 = \theta_2^* \) is a solution, \( \theta_2 = \pi - \theta_2^* \) is also a solution.

Corresponding to each solution set of \((\theta_1, \theta_2)\), Eqs. (2.49) and (2.50) produce a unique solution of \( \theta_{234} \):

\[
\theta_{234} = \text{Atan2} \left[ -u_z/c\theta_5, (u_x c\theta_1 + u_y s\theta_1)/c\theta_5 \right]. \tag{2.57}
\]

Next, we solve Eqs. (2.52) and (2.53) for \( \theta_2 \) and \( \theta_3 \). Equations (2.52) and (2.53) can be written
\begin{align}
    a_2 \cos \theta_2 + a_3 \cos \theta_{23} &= k_1, \\
    a_2 \sin \theta_2 + a_3 \sin \theta_{23} &= k_2.
\end{align}

where \( k_1 = q_x \cos \theta_1 + q_y \sin \theta_1 - a_1 + d_5 \cos \theta_{234} \) and \( k_2 = -q_z + d_1 - d_5 \cos \theta_{234} \).

Summing the squares of Eqs. (2.58) and (2.59) yields

\begin{align}
    a_2^2 + a_3^2 + 2a_2a_3 \cos \theta_3 &= k_1^2 + k_2^2. \tag{2.60}
\end{align}

Hence

\[ \theta_3 = \cos^{-1} \left( \frac{k_1^2 + k_2^2 - a_2^2 - a_3^2}{2a_2a_3} \right). \tag{2.61} \]

and there are two solutions of \( \theta_3 \); that is, if \( \theta_3 = \theta_3^* \) is a solution, \( \theta_3 = -\theta_3^* \) is also a solution.

Once \( \theta_3 \) is known, we can solve \( \theta_2 \) by expanding Eqs. (2.58) and (2.59) as follows:

\begin{align}
    (a_2 + a_3 \cos \theta_3) \cos \theta_2 - (a_3 \sin \theta_3) \sin \theta_2 &= k_1, \tag{2.62} \\
    (a_3 \sin \theta_3) \cos \theta_2 + (a_2 + a_3 \sin \theta_3) \sin \theta_2 &= k_2. \tag{2.63}
\end{align}

Solving Eqs. (2.62) and (2.63) for \( \cos \theta_2 \) and \( \sin \theta_2 \) yields

\begin{align*}
    \cos \theta_2 &= \frac{k_1 (a_2 + a_3 \cos \theta_3) + k_2 a_3 \sin \theta_3}{a_2^2 + a_3^2 + 2a_2 a_3 \cos \theta_3}, \\
    \sin \theta_2 &= \frac{-k_1 a_3 \sin \theta_3 + k_2 (a_2 + a_3 \cos \theta_3)}{a_2^2 + a_3^2 + 2a_2 a_3 \cos \theta_3}.
\end{align*}

Hence, corresponding to each solution set of \( (\theta_1, \theta_3, \theta_3, \theta_{234}) \), we obtain a unique solution of \( \theta_2 \):

\[ \theta_2 = \text{Atan2}(\sin \theta_2, \cos \theta_2). \tag{2.64} \]

Finally, \( \theta_4 \) is obtained by

\[ \theta_4 = \theta_{234} - \theta_2 - \theta_3. \tag{2.65} \]

We conclude that corresponding to each given end-effector location, there are at most eight inverse kinematics solutions.
\[ k_1, \quad (2.58) \]
\[ k_2. \quad (2.59) \]
\[ k_2 = -q_z + d_1 - d_3 c \theta_{234}. \quad (2.59) \]

Yields
\[ k_1^2 + k_2^2. \quad (2.60) \]
\[ \frac{k_1^2 - a_3^2}{3}. \quad (2.61) \]

\[ \theta_3^* \text{ is a solution, } \theta_3 = -\theta_3^* \text{ is } \]

Solving Eqs. (2.58) and (2.59) as
\[ s \theta_2 = k_1, \quad (2.62) \]
\[ s \theta_2 = k_2. \quad (2.63) \]
\[ \text{yields} \]
\[ \frac{k_2 c_1 s \theta_3}{a_3 c \theta_3} + \frac{a_3 c \theta_3}{a_3 c \theta_3}. \]
\[ f (\theta_1, \theta_3, \theta_5, \theta_{234}), \text{ we obtain a} \]
\[ \theta_2). \quad (2.64) \]
\[ \theta_3. \quad (2.65) \]

ven end-effector location, there

2.6.3 Position Analysis of the Fanuc S-900W Robot

Figure 2.8 shows a 6-dof manipulator manufactured by Fanuc. In this manipulator, the first joint axis points up vertically along the z_0-axis, the second joint axis is perpendicular to the first joint axis with a small offset distance \( a_1 = OA \), the third joint axis is parallel to the second with an offset distance \( a_2 = AB \), and the fourth joint axis is perpendicular to the third joint axis with a small offset distance \( a_3 = BC \). In addition, the last three joint axes intersect one another perpendicularly in sequence at a common point \( P \), which is \( d_4 \) distance away from point \( C \). This robot belongs to a special class of manipulators where the last three joint axes intersect at the wrist center. The kinematics problem for this type of manipulators can be partitioned into two subchains: one associated with the first three moving links and the other with the last three moving links. That is, in solving the inverse kinematics problem, the position of the wrist center can be solved independently of the orientation part, therefore reducing the complexity of the problem.

![Diagram of Fanuc S-900W robot](image-url)
TABLE 2.4. D-H Parameters of the Fanuc S-900W Manipulator

<table>
<thead>
<tr>
<th>Joint $i$</th>
<th>$\alpha_i$</th>
<th>$a_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi/2$</td>
<td>$a_1$</td>
<td>0</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$a_2$</td>
<td>0</td>
<td>$\theta_2$</td>
</tr>
<tr>
<td>3</td>
<td>$\pi/2$</td>
<td>$a_3$</td>
<td>0</td>
<td>$\theta_3$</td>
</tr>
<tr>
<td>4</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>$d_4$</td>
<td>$\theta_4$</td>
</tr>
<tr>
<td>5</td>
<td>$\pi/2$</td>
<td>0</td>
<td>0</td>
<td>$\theta_5$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>$d_6$</td>
<td>$\theta_6$</td>
</tr>
</tbody>
</table>

We note that this manipulator employs a four-bar linkage to drive the third joint. The four-bar linkage simply transmits the motion of the third motor mounted on the waist to the third joint. Otherwise, it has no effect on the kinematics of the manipulator. In the following analysis we neglect the effect of the four-bar linkage and treat the manipulator as a serial manipulator.

Using the coordinate systems established in Fig. 2.8, the corresponding link parameters are listed in Table 2.4. Substituting the D-H link parameters into Eq. (2.2), we obtain the D-H transformation matrices:

$$^0A_1 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & a_1 c\theta_1 \\ s\theta_1 & 0 & -c\theta_1 & a_1 s\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.66)$$

$$^1A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.67)$$

$$^2A_3 = \begin{bmatrix} c\theta_3 & 0 & s\theta_3 & a_3 c\theta_3 \\ s\theta_3 & 0 & -c\theta_3 & a_3 s\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.68)$$

$$^3A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.69)$$

$$^4A_5 = \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.70)$$

The loop-closure equation Eqs. (2.66), (2.67), and (2.68) can be written as:

$$^0A_6 = ^0A_1 ^1A_2 ^2A_3$$

$$^0A_6 = \begin{bmatrix} c\theta_1 c\theta_23 & s\theta_1 & s\theta_1 c\theta_23 & -c\theta_1 \\ s\theta_1 c\theta_23 & c\theta_1 & c\theta_1 c\theta_23 & s\theta_1 \\ s\theta_23 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & 0 & 0 & 0 \\ s\theta_23 c\theta_5 c\theta_6 + c\theta_4 s\theta_6 & 0 & 0 & 0 \end{bmatrix}$$

Next, we multiply Eqs. (2.69) and get

$$^5A_6 = ^3A_4 ^4A_5 ^5A_6$$

Hence the resulting transformation

$$^0A_6 = \begin{bmatrix} c\theta_1 c\theta_23 & s\theta_1 & s\theta_1 c\theta_23 & -c\theta_1 \\ s\theta_1 c\theta_23 & c\theta_1 & c\theta_1 c\theta_23 & s\theta_1 \\ s\theta_23 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & 0 & 0 & 0 \\ s\theta_23 c\theta_5 c\theta_6 + c\theta_4 s\theta_6 & 0 & 0 & 0 \end{bmatrix}$$

where $^0A_6$ describes the end-effector location.

Substituting Eqs. (2.73) and the corresponding link parameters as follows:

$$u_x = c\theta_1 |c\theta_23| c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6$$

$$u_y = s\theta_1 |c\theta_23| c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6$$

$$u_z = s\theta_23 (c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6)$$
The end-effector location is given by

\[
0_A = \begin{bmatrix}
  u_x & u_x & w_x & q_x \\
  u_y & u_y & w_y & q_y \\
  u_z & u_z & w_z & q_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]  

(2.72)

The loop-closure equation is obtained in two steps. First, we multiply Eqs. (2.66), (2.67), and (2.68):

\[
0_A = 0_A 1_A 2_A 3
\]

Next, we multiply Eqs. (2.69), (2.70), and (2.71):

\[
3_A = 3_A 4_A 5_A 6
\]

Hence the resulting transformation matrix is given by

\[
0_A = 0_A 3_A 6,
\]

(2.75)

where \(0_A\) describes the end effector location.

Substituting Eqs. (2.73) and (2.74) into (2.75) yields the elements of \(0_A\) as follows:

\[
u_x = c_{\theta_1} (c_{\theta_4} c_{\theta_5} c_{\theta_6} - s_{\theta_4} s_{\theta_6}) + s_{\theta_1} (s_{\theta_4} c_{\theta_5} c_{\theta_6} + c_{\theta_4} s_{\theta_6}),
\]

(2.76)

\[
u_y = s_{\theta_1} (c_{\theta_4} c_{\theta_5} c_{\theta_6} - s_{\theta_4} s_{\theta_6}) - s_{\theta_2} s_{\theta_5} c_{\theta_6} - s_{\theta_2} s_{\theta_5} c_{\theta_6} + c_{\theta_1} (s_{\theta_4} c_{\theta_5} c_{\theta_6} + c_{\theta_4} s_{\theta_6}),
\]

(2.77)

\[
u_z = s_{\theta_2} (c_{\theta_4} c_{\theta_5} c_{\theta_6} - s_{\theta_4} s_{\theta_6}) + c_{\theta_2} s_{\theta_5} c_{\theta_6},
\]

(2.78)
\[ v_x = c_{ \theta_1} [-c_{ \theta_23} (c_{ \theta_4} c_{ \theta_5} s_{ \theta_6} + s_{ \theta_4} c_{ \theta_6}) + s_{ \theta_23} s_{ \theta_5} s_{ \theta_6}] \\
+ s_{ \theta_1} (-s_{ \theta_4} c_{ \theta_5} s_{ \theta_6} + c_{ \theta_4} c_{ \theta_6}), \\
v_y = s_{ \theta_1} [-c_{ \theta_23} (c_{ \theta_4} c_{ \theta_5} s_{ \theta_6} + s_{ \theta_4} c_{ \theta_6}) + s_{ \theta_23} s_{ \theta_5} s_{ \theta_6}] \\
- c_{ \theta_1} (-s_{ \theta_4} c_{ \theta_5} s_{ \theta_6} + c_{ \theta_4} c_{ \theta_6}), \\
v_z = -s_{ \theta_{23}} (c_{ \theta_4} c_{ \theta_5} s_{ \theta_6} + s_{ \theta_4} c_{ \theta_6}) - c_{ \theta_{23}} s_{ \theta_5} s_{ \theta_6}, \\
w_x = c_{ \theta_1} (c_{ \theta_{23}} c_{ \theta_4} s_{ \theta_5} + s_{ \theta_{23}} c_{ \theta_5}) + s_{ \theta_1} s_{ \theta_4} s_{ \theta_5}, \\
w_y = s_{ \theta_1} (c_{ \theta_{23}} c_{ \theta_4} s_{ \theta_5} + s_{ \theta_{23}} c_{ \theta_5}) - c_{ \theta_1} s_{ \theta_4} s_{ \theta_5}, \\
w_z = s_{ \theta_{23}} c_{ \theta_4} s_{ \theta_5} - c_{ \theta_{23}} c_{ \theta_5}, \\
q_x = c_{ \theta_1} [a_1 + a_2 c_{ \theta_2} + a_3 c_{ \theta_23} + d_4 s_{ \theta_{23}} + d_6 (c_{ \theta_{23}} c_{ \theta_4} s_{ \theta_5} + s_{ \theta_{23}} c_{ \theta_5})] \\
+ d_6 s_{ \theta_1} s_{ \theta_4} s_{ \theta_5}, \\
q_y = s_{ \theta_1} [a_1 + a_2 c_{ \theta_2} + a_3 c_{ \theta_23} + d_4 s_{ \theta_{23}} + d_6 (c_{ \theta_{23}} c_{ \theta_4} s_{ \theta_5} + s_{ \theta_{23}} c_{ \theta_5})] \\
- d_6 c_{ \theta_1} s_{ \theta_4} s_{ \theta_5}, \\
q_z = a_2 s_{ \theta_2} + a_3 s_{ \theta_{23}} - d_4 c_{ \theta_{23}} + d_6 (s_{ \theta_{23}} c_{ \theta_4} s_{ \theta_5} - c_{ \theta_{23}} c_{ \theta_5}). \\
\]

Although the equations above can be used to solve the inverse kinematics, they are highly nonlinear and difficult to solve. In what follows we present a more efficient method of solution by separating the wrist-center-position problem from the orientation problem.

(a) Wrist Center Position. Note that the last three joint axes intersect at the wrist center point \( P \) as shown in Fig. 2.8. Hence rotations of the last three joints do not affect the position of \( P \).

Figure 2.9 shows the end-effector coordinate system \( (x_6, y_6, z_6) \), the wrist center \( P \), and the vector relation between them.

The wrist center position with respect to and expressed in the end-effector coordinate system is

\[ \hat{0}p = \overrightarrow{OP} = [0, 0, -d_6, 1]^T. \tag{2.76} \]

The wrist center position with respect to and expressed in the base coordinate system is

\[ \hat{0}p = \overrightarrow{OP} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} q_x - d_6 w_z \\ q_y - d_6 w_z \\ q_z - d_6 w_z \\ 1 \end{bmatrix}. \tag{2.77} \]
to solve the inverse kinematics, we have. In what follows we present the wrist-center-position

last three joint axes intersect at

Hence rotations of the last three joints show the end-effector coordinate system expressed in the end-effector

and expressed in the base coordinate

and the vector relation between

Hence, given the end-effector location, we can find the position of the wrist center point \( P \) with respect to the base coordinate system. Furthermore, we observe from Fig. 2.8 that the position of the wrist center \( P \) with respect to the link 3 coordinate system is given by

\[
^3\mathbf{p} = \mathbf{C}^P = [0, 0, d_4, 1]^T.
\] (2.78)

Transforming \( ^3\mathbf{p} \) into the base coordinate system, we obtain

\[
{}^0\mathbf{p} = {}^0A_3{}^3\mathbf{p}.
\] (2.79)

Equation (2.79) consists of three scalar equations in three unknowns. Hence the position and orientation of the inverse kinematics problem are decoupled. Theoretically, we can solve Eq. (2.79) for the three joint angles. In what follows we take a simpler approach. Multiplying both sides Eq. (2.79) by the inverse of \( {}^0A_1 \), we obtain

\[
({}^0A_1)^{-1}{}^0\mathbf{p} = {}^1A_3{}^3\mathbf{p}.
\] (2.80)

Substituting Eqs. (2.66) through (2.68) into (2.80) yields

\[
p_x c\theta_1 + p_y s\theta_1 - a_4 = a_2 c\theta_2 + a_3 c\theta_3 + d_4 s\theta_23,
\] (2.81)

\[
p_z = a_2 s\theta_2 + a_3 s\theta_3 - d_4 c\theta_23.
\] (2.82)
where \( p_x, p_y, \) and \( p_z \) are given by Eq. (2.77).

A solution for \( \theta_1 \) is found immediately by solving Eq. (2.83).

\[
\theta_1 = \arctan \frac{p_y}{p_x},
\]

(2.84)

Hence there are two solutions of \( \theta_1 \). Specifically, if \( \theta_1 = \theta_1^+ \) is a solution, \( \theta_1 = \theta_1^+ + \pi \) is also a solution, where \( \pi \geq \theta_1^+ \geq 0 \). We call \( \theta_1 = \theta_1^+ \) the front-reach solution and \( \theta_1 = \theta_1^+ + \pi \) the back-reach solution. Because of the four-bar linkage and other mechanical constraints, the back-reach solution is physically impossible.

An observation of the kinematic structure reveals that the distance between point A and the wrist center \( P \) is independent of \( \theta_1 \) and \( \theta_2 \), which implies that these two variables can be eliminated simultaneously. Summing the squares of Eqs. (2.81), (2.82), and (2.83), gives

\[
\kappa_1 s\theta_3 + \kappa_2 c\theta_3 = \kappa_3,
\]

(2.85)

where \( \kappa_1 = 2a_2d_4, \kappa_2 = 2a_2a_3, \) and \( \kappa_3 = p_x^2 + p_y^2 + p_z^2 - 2p_x a_1 c\theta_1 - 2p_y a_1 s\theta_1 - a_1^2 - a_2^2 - a_3^2 - d_2^2. \)

We can convert Eq. (2.85) into a polynomial by making use of the following trigonometric identities:

\[
c\theta_3 = \frac{1 - t_3^2}{1 + t_3^2} \quad \text{and} \quad s\theta_3 = \frac{2t_3}{1 + t_3^2}, \quad \text{where} \quad t_3 = \tan \frac{\theta_3}{2}.
\]

Substituting the trigonometric identities above into Eq. (2.85) yields

\[
(\kappa_3 + \kappa_2)t_3^2 - 2\kappa_1 t_3 + (\kappa_3 - \kappa_2) = 0.
\]

(2.86)

Hence

\[
\frac{\theta_3}{2} = \arctan \frac{\kappa_1 \pm \sqrt{\kappa_1^2 + \kappa_2^2 - \kappa_3^2}}{\kappa_3 + \kappa_2}.
\]

(2.87)

Equation (2.86) yields (1) two real roots if \( \kappa_1^2 + \kappa_2^2 - \kappa_3^2 > 0 \), (2) one double root if \( \kappa_1^2 + \kappa_2^2 - \kappa_3^2 = 0 \), and (3) no real roots if \( \kappa_1^2 + \kappa_2^2 - \kappa_3^2 < 0 \). When Eq. (2.86) yields a double root, the arm is either in a fully stretched or a folded-back configuration. On the other hand, if Eq. (2.86) yields no real roots, the position is not reachable. Figure 2.10 shows two different arm configurations, corresponding to the two solutions of \( \theta_1 \), only two are physically possible.
by solving Eq. (2.83).

\[
\theta_2 = \frac{p_x}{2} \tag{2.84}
\]

Specifically, if \( \theta_1 = \theta_1^* \) is a solution, \( \geq \theta_1^* \geq 0 \). We call \( \theta_1 = \theta_1^* \) the back-reach solution. Because of the constraints, the back-reach solution is revealed that the distance between \( \theta_1 \) and \( \theta_2 \), which implies that simultaneously. Summing the squares

\[
\kappa_1^2 - \kappa_2^2 = \kappa_3, \tag{2.85}
\]

\[
p_x^2 + p_y^2 + p_z^2 - 2p_xa_1a_\theta_1 -
\]

\[
= \kappa_3, \tag{2.86}
\]

\[
\frac{-p_x^2 - \kappa_2^2 - \kappa_3^2}{\kappa_2}. \tag{2.87}
\]

Once \( \theta_1 \) and \( \theta_3 \) are known, \( \theta_2 \) can be obtained by back substitution. Expanding Eqs. (2.81) and (2.82), we obtain

\[
\mu_1 c\theta_2 + \nu_1 s\theta_2 = \gamma_1, \tag{2.88}
\]
\[
\mu_2 c\theta_2 + \nu_2 s\theta_2 = \gamma_2, \tag{2.89}
\]

where

\[
\mu_1 = a_2 + a_3c\theta_1 + d_4s\theta_3, \]
\[
\nu_1 = -a_3s\theta_1 + d_4c\theta_3, \]
\[
\gamma_1 = p_x c\theta_1 + p_y s\theta_1 - a_1, \]
\[
\mu_2 = a_3s\theta_0 - d_4c\theta_3, \]
\[
\nu_2 = a_2 + a_3c\theta_1 + d_4s\theta_3, \]
\[
\gamma_2 = p_z. \]

Therefore, we can solve Eqs. (2.88) and (2.89) for \( c\theta_2 \) and \( s\theta_2 \). Once \( s\theta_2 \) and \( c\theta_2 \) are found, a unique value of \( \theta_2 \) is obtained by taking

\[
\theta_2 = \text{Atan2}(s\theta_2, c\theta_2). \tag{2.90}
\]

We conclude that given the wrist center position, mathematically there are at most four possible arm configurations, but due to the mechanical limits, only two are physically possible.
(b) End-Effector Orientation. Once $\theta_1$, $\theta_2$, and $\theta_3$ are solved, $\theta_3$ is completely known. The remaining joint angles can be found by multiplying both sides of Eq. (2.75) by $(A_3)^{-1}$:

$$A_6 = (A_3)^{-1} A_6.$$  

(2.91)

We note that the elements on the right-hand side of Eq. (2.91) are known, and only the rotation part of Eq. (2.91) is needed for computation of the last three joint angles. The rotation matrices $R_3$ and $R_6$ are given by the upper $3 \times 3$ submatrices of Eqs. (2.73) and (2.74), respectively.

Equating the $3 \times 3$ element of Eq. (2.91) yields

$$\theta_5 = \cos^{-1} r_{33},$$  

(2.92)

where $r_{33} = w_x c \theta_1 s \theta_2 + w_y s \theta_1 c \theta_2 - w_z c \theta_23$. Hence, corresponding to each solution set of $\theta_1$, $\theta_2$, and $\theta_3$, Eq. (2.92) yields (1) two real roots if $|r_{33}| < 1$, and (2) $\theta_5 = 0$ or $\pi$ if $|r_{33}| = 1$. When $\theta_5 = 0$ or $\pi$, the sixth joint axis, $z_5$, is in line with the fourth joint axis, $z_3$, and the wrist is said to be in a singular configuration. The condition $|r_{33}| > 1$ cannot physically arise.

Assuming that $s \theta_5 \neq 0$, we can solve $\theta_4$ and $\theta_6$ as follows. Equating the $1 \times 3$ element of Eq. (2.91) yields

$$c \theta_4 = \frac{w_x c \theta_1 c \theta_23 + w_y s \theta_1 c \theta_23 + w_z s \theta_23}{s \theta_5}.$$  

(2.93)

Equating the $2 \times 3$ element of Eq. (2.91) yields

$$s \theta_4 = \frac{w_x s \theta_1 - w_y c \theta_1}{s \theta_5}.$$  

(2.94)

Hence, corresponding to each solution set of $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_5$, Eqs. (2.93) and (2.94) yield a unique solution of $\theta_4$:

$$\theta_4 = \text{Atan2}(s \theta_4, c \theta_4).$$  

(2.95)

Similarly, equating the $3 \times 1$ element of Eq. (2.91) yields

$$c \theta_6 = \frac{u_x c \theta_1 s \theta_23 + u_y s \theta_1 s \theta_23 - u_z c \theta_23}{s \theta_5}.$$  

(2.96)

Equating the $3 \times 2$ element of Eq. (2.91) yields

$$s \theta_6 = \frac{v_x c \theta_1 s \theta_23 + v_y s \theta_1 s \theta_23 - v_z c \theta_23}{s \theta_5}.$$  

(2.97)

Hence, corresponding to each solution set of $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_5$, yields a unique solution.

We conclude that corresponding to each solution set of $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_5$, there are two possible upper arm configurations, a singular configuration. However, due to mechanical limitations, only one configuration is physically realizable. When $\theta_5 = 0$ or $\pi$, the upper arm configuration is physically realizable.

2.6.4 Tsai and Morgan's Inverse Kinematics

In this section we outline a solution to the inverse kinematics problem for a serial manipulator, who reduced the problem to the inverse kinematics problem of a planar mechanism equivalent to a planar mechanism. For such a singular configuration, there are two possible upper arm configurations, and (2.97) yield a unique solution of $\theta_6$.

$$\theta_6 = \text{Atan2}(s \theta_6, c \theta_6).$$  

(2.98)

For convenience, we introduced the vector $e$ of the $z_5$-axis of the end-effector coordinate system. Using the vector equation can be written as

$$0 A_1 = A_2.$$  

(2.99)

These two vectors can be expressed as $e = \begin{bmatrix} 0, 0, 1 \end{bmatrix}^T$ and $e = \begin{bmatrix} 0, 0, 1 \end{bmatrix}^T$. Using the vector equation can be written as

$$e = A_6 A_5 A_2.$$  

(2.100)
and $\theta_3$ are solved, $^0A_3$ can be found by multiplying both sides of Eq. (2.91) by $^{0}A_3$.

Equating the right side of Eq. (2.91) are known, and for computation of the last three $^3R_6$ are given by the upper $3 \times 3$ yields

(2.92)

Hence, corresponding to each solution set of $\theta_1, \theta_2, \theta_3, \theta_4, \text{ and } \theta_5$, Eqs. (2.93) and (2.97) yield a unique solution of $\theta_6$:

(2.98)

We conclude that corresponding to each solution set of the first three joint angles, there are two possible wrist configurations. Since there are four possible upper arm configurations, a total of eight manipulator postures are possible. However, due to mechanical limits, fewer than eight manipulator postures are physically realizable. When $s\theta_6 = 0$, Eqs. (2.93) through (2.98) degenerate. For such a singular condition, only the sum or difference of $\theta_4$ and $\theta_6$ can be computed.

2.6.4 Tsai and Morgan's Solution

In this section we outline a solution method developed by Tsai and Morgan (1985), who reduced the problem to a system of four equations and then employed a numerical method, known as the homotopy method, to find all solutions to the inverse kinematics of a general 6R manipulator. They also derived closed-form solutions for manipulators in which three consecutive joint axes either intersect at a common point or are parallel to one another.

Figure 2.11 shows a general 6R manipulator where point $Q$ denotes the origin and $u$, $v$, and $w$ denote three orthogonal unit vectors of the end effector coordinate system. Using the Denavit–Hartenberg method, a loop-closure equation can be written as

(2.99)

For convenience, we introduce a position vector $p$ of the origin and a unit vector $e$ of the $z_5$-axis of the fifth coordinate system as shown in Fig. 2.11. These two vectors can be expressed in the fifth coordinate system as $^5p = [0, 0, 0, 1]^T$ and $^5e = [e_x, e_y, e_z, 0]^T$, or in the fixed coordinate system as $p = ^0p = [p_x, p_y, p_z, 1]^T$ and $e = ^0e = [e_x, e_y, e_z, 0]^T$. Since both the point $P$ and the vector $e$ are attached to the end effector, $p$ and $e$ can be computed from the given end-effector location as follows:

(2.100)
Equations (2.100) and (2.101) imply that once the end-effector location is given, the point \( P \) and the direction of the \( z_5 \)-axis can be found. The transformation between \( s_p \) and \( s_0 \) and between \( e_5 \) and \( e_0 \) can be written as:

\[
0_p = 0_{A_6} 6_{A_5} \begin{bmatrix} e_x \\ e_y \\ e_z \\ 0 \end{bmatrix} = \begin{bmatrix} v_x s_6 + w_x c_6 \\ v_y s_6 + w_y c_6 \\ v_z s_6 + w_z c_6 \\ 0 \end{bmatrix}.
\]

(2.101)

Equations (2.100) and (2.101) imply that once the end-effector location is given, the point \( P \) and the direction of the \( z_5 \)-axis can be found.

The transformation between \( e_5 \) and \( e_0 \) and between \( e_5 \) and \( e_0 \) can be written as:

\[
0_e = 0_{A_1} ^1A_2 3_{A_3} 4_{A_4} 5_{A_5} e_5.
\]

(2.102)

(2.103)

To simplify the analysis, we multiply both sides of Eqs. (2.102) and (2.103) by \( (0_{A_1} ^1A_2)^{-1} \). The resulting equations can be written:

\[
^2p = ^2p',
\]

(2.104)

\[
^2e = ^2e'.
\]

(2.105)

where

\[
\begin{align*}
^2p &= ^2p' \\
^2e &= ^2e'
\end{align*}
\]

are the position vectors of \( P \) and \( E \) in the \((x_2, y_2, z_2)\) coordinate system.

Equations (2.104) and (2.105) are already free of the variable \( \theta_6 \). However, only \( \theta_1, \ldots, \theta_5 \) are independent, because the \( \alpha_i \) is a unit vector. Hence there are five knowns, \( \theta_1, \ldots, \theta_5 \). The \( x \) and \( z \) coordinates of the \( x \) and \( z \) units in the sines and cosines of this approach, \( \theta_6 \), does not appear in the third-degree polynomials, while the \( x \) and \( z \) polynomials in the sines and cosines of this approach, \( \theta_6 \), does not appear in the system of equations above.

Elimination of \( \theta_3 \). First, we note that Eqs. (2.104) and (2.105) are already free of \( \theta_3 \) and \( \theta_4 \). The resulting equations can be written:

\[
\begin{align*}
h_x &= a_3 c_6 \theta_2 - h_y c_6 \\
n_x &= a_3 c_6 \theta_2 - h_y c_6 \\
g_x &= a_2 c_6 \theta_2 + g_y c_6 \\
g_z &= a_2 c_6 \theta_2 + g_y c_6 \\
h_z &= p_x c_6 \theta_1 + p_y c_6 \theta_1 \\
h_z &= p_x c_6 \theta_1 + p_y c_6 \theta_1 \\
m_x &= c_6 \theta_3 s_6.
\end{align*}
\]

where
The end-effector location is found.

\[
\begin{align*}
\mathbf{p} &= A_3 A_4 A_5 p, \\
\mathbf{e} &= A_3 A_4 A_5 e,
\end{align*}
\]

are the position vectors of \( P \) and the direction of the \( z_5 \)-axis with reference to the \((x_2, y_2, z_2)\) coordinate system.

Equations (2.104) and (2.105) constitute a set of six scalar equations free of the variable \( \theta_6 \). However, only two of the three scalar equations in Eq. (2.105) are independent, because the components of \( e \) must satisfy the condition of a unit vector. Hence there are only five independent equations in five unknowns, \( \theta_1, \ldots, \theta_5 \). The \( x \) and \( y \) components in Eqs. (2.104) and (2.105) are third-degree polynomials, while the \( z \)-component is a second-degree polynomial in the sines and cosines of five joint angles. We note that by using this approach, \( \theta_6 \) does not appear in the system of equations and therefore reduces the complexity of the problem. In the following we eliminate \( \theta_3 \) from the system of equations above.

**Elimination of \( \theta_3 \).** First, we notice that both \( z \)-components of Eqs. (2.104) and (2.105) are already free of the variable \( \theta_3 \). Expanding the \( z \)-components of Eqs. (2.104) and (2.105), yields

\[
\begin{align*}
&h_z \cos \theta_2 - h_3 \sin \theta_2 \theta_1 = -h_3 \cos \theta_2 + d_3 \cos \theta_2, \\
&n_z \sin \theta_2 \theta_1 - n_3 \cos \theta_2 \theta_1 = -n_3 \sin \theta_2 + m_3 \cos \theta_2 \theta_1
\end{align*}
\]

where

\[
\begin{align*}
g_z &= a_3 \cos \theta_3 + a_4, \\
g_3 &= -a_3 \cos \theta_3 \theta_5 + d_3 \sin \theta_4, \\
g_e &= a_3 \sin \theta_3 \theta_5 + d_3 \cos \theta_4 + d_4, \\
h_z &= p_x \cos \theta_1 + p_y \sin \theta_1 - a_1, \\
h_3 &= -p_x \cos \theta_1 \theta_5 + p_y \sin \theta_1 \theta_5 + (p_z - d_1) \cos \theta_5, \\
h_e &= p_x \sin \theta_1 \theta_5 - p_y \cos \theta_1 \theta_5 + (p_z - d_1) \sin \theta_5, \\
m_z &= a_5 \sin \theta_5, \\
m_3 &= a_5 \sin \theta_5 \theta_1.
\end{align*}
\]
POSITION ANALYSIS OF SERIAL MANIPULATORS

\[ m_y = c \alpha_4 s \alpha_5 c \theta_3 + s \alpha_4 c \alpha_5, \]

\[ m_z = -s \alpha_4 s \alpha_5 c \theta_3 + c \alpha_4 c \alpha_5, \]

\[ n_x = e_x c \theta_1 + e_y s \theta_1, \]

\[ n_y = -e_x c \alpha_1 s \theta_1 + e_y c \alpha_1 c \theta_1 + e_z s \alpha_1, \]

\[ n_z = e_x s \alpha_1 s \theta_1 - e_y s \alpha_1 c \theta_1 + e_z c \alpha_1. \]

A third equation that is free of \( \theta_1 \) is obtained by performing the dot product \( \mathbf{p} \cdot \mathbf{e} = \mathbf{p}' \cdot \mathbf{e}' \). Substituting Eqs. (2.104) and (2.105) into the dot product and simplifying, we obtain

\[ a_2 n_y s \theta_2 + a_2 n_x c \theta_2 + (a_3 m_z + d_3 m_x s \alpha_3) s \theta_4 + (a_3 m_x - d_3 m_y s \alpha_3) c \theta_4 \]

\[ = -a_1 n_x - d_2 n_y - a_4 m_x - m_z (d_3 c \alpha_3 + d_4) + k_1, \]

where \( k_1 = -a_3 c \alpha_5 + p_x e_x + p_y e_y + (p_z - d_1) e_z \).

A fourth equation that is free of \( \theta_1 \) is obtained by equating the sum of the squares of the \( x \), \( y \), and \( z \) components on both sides of Eq. (2.104). Expanding \((\mathbf{p}^p)^2 = (\mathbf{p}')^2\) yields

\[ a_2 h_y s \theta_2 + a_2 h_x c \theta_2 + (a_3 g_y + d_3 g_x s \alpha_3) s \theta_4 + (a_3 g_x - d_3 g_y s \alpha_3) c \theta_4 \]

\[ = -a_1 h_x - d_2 h_y - a_4 g_x - g_z (d_3 c \alpha_3 + d_4) + k_2, \]

where

\[ k_2 = 0.5[p_x^2 + p_y^2 + (p_z - d_1)^2 - a_1^2 + a_2^2 + a_3^2 - a_4^2 - a_5^2 + a_6^2 - a_7^2 - a_8^2]. \]

Equations (2.106) through (2.109) represent a system of four second-degree polynomials in the sines and cosines of four joint angles. We may consider \( \sin \theta_i \) and \( \cos \theta_i \) as two independent variables and add the following trigonometric identities as supplementary equations of constraint:

\[ \sin^2 \theta_i + \cos^2 \theta_i = 1, \quad \text{for} \quad i = 1, 2, 4, 5. \]

In this way, we obtain a system of eight second-degree polynomials in eight variables. Tsai and Morgan employed a continuation method to solve the system of equations above and showed that the most general 6-dof, \( 6R \) robot has at most 16 significant solutions. See Appendix A for more details.

The system of equations will decouple when any three consecutive joint axes either intersect at a common point or are parallel to one another. For these special geometries, closed-form solutions can be derived. In what follows we illustrate the decoupling by solving the inverse kinematics of two special cases that are most commonly implemented in industrial robots. Other special cases can be derived by applying the kinematic inversions.

(a) Last Three Joint Axes

The last three joint axes intersect identically. Substituting these \( \theta_1 \),

\[ h_y s \theta_2 \]

\[ h_y s \theta_2 \]

provided that \( s \alpha_2 \neq 0 \) and \( a_2 \neq 0 \),

\[ \mu_1 = -h_z c \alpha_2 + d_2 c \alpha_3 + d_4, \]

\[ \mu_2 = -a_1 h_x - d_2 h_z - d_3 d_1 \]

\[ + 0.5(p_x^2 + p_y^2 + (p_z - d_1)^2). \]

Equations (2.111) and (2.112) hence are completely decoupled \( \theta_2 \); we sum the squares of Eqs.

\[ h_x^2 + h_y^2 \]

Equation (2.113) contains a fourth-degree polynomial in \( h_z \) with \((1 - t_1^2)/(1 + t_1^2)\), where \( t_1 \) is the effector position, there are at most one solution of \( \theta_2 \) can be found simultaneously for \( s \theta_2 \) and \( c \theta_2 \), i.e., \( t_1 \) is a function. Following that, a unique solution of \( \theta_2 \) can be found by following Eq. (2.113). Hence we conclude that there exists only one solution for \( \theta_2 \).

(b) Joint Axes 2, 3, and 4

Third, and fourth joint axes are parallel. Further, since the common axes and between the third and fourth axes always define these two common values, Eqs. (2.106) and (2.109) hold.

\[ s \alpha_4 s \alpha_5 \]
led by performing the dot product
and 

\( \text{Equation (2.105)} \)

\[ \begin{align*}
\dot{s}_4 + (a_3 s_x - d_3 s_3 s_\alpha_3) c_\theta_4 \\
+ d_4 + k_1,
\end{align*} \]

provided that \( s_\alpha_2 \neq 0 \) and \( a_2 \neq 0 \), where

\[ \begin{align*}
\mu_1 &= -h_4 c_\alpha_2 + d_3 c_\alpha_1 + d_4 c_\alpha_3, \\
\mu_2 &= -a_1 h_4 - d_2 h_2 - d_3 d_4 c_\alpha_3 \\
&\quad + 0.5(p_x^2 + p_y^2 + (p_z - d_1)^2 - a_1^2 + a_2^2 + d_2^2 - a_3^2 - d_3^2 - d_4^2).
\end{align*} \]

Equations (2.111) and (2.112) contain only two unknown variables and hence are completely decoupled from Eqs. (2.107) and (2.108). To eliminate \( \theta_2 \), we sum the squares of Eqs. (2.111) and (2.112).

\[ \begin{align*}
h_x^2 + h_y^2 &= (\mu_1/s_\alpha_2)^2 + (\mu_2/a_2)^2.
\end{align*} \]

Equation (2.113) contains only one variable, \( \theta_1 \). We may convert it into a fourth-degree polynomial in \( t_1 \) by replacing \( s_\theta_1 \) with \( 2t_1/(1 + t_1^2) \) and \( c_\theta_1 \) with \( (1 - t_1^2)/(1 + t_1^2) \), where \( t_1 = \tan(\theta_1/2) \). Hence, for each given end-effector position, there are at most four real solutions of \( \theta_1 \). Once \( \theta_1 \) is found, a unique solution of \( \theta_2 \) can be obtained by solving Eqs. (2.111) and (2.112) simultaneously for \( s_\theta_2 \) and \( c_\theta_2 \), and then applying the two-argument arctangent function. Following that, a unique solution of \( \theta_3 \) can be found by solving the two scalar equations associated with the \( x \) and \( y \) components of Eq. (2.104).

Corresponding to each solution set of \( (\theta_1, \theta_2, \theta_3) \), two sets of \( (\theta_4, \theta_5, \theta_6) \) can be found by following the procedure outlined in the earlier example. Hence we conclude that there are at most eight possible solutions sets (manipulator postures).

**(b) Joint Axes 2, 3, and 4 Parallel to One Another.** When the second, third, and fourth joint axes are parallel to one another, \( a_2 = a_3 = 0 \) identically. Further, since the common normals between the second and third joint axes and between the third and fourth joint axes are indeterminate, we can always define these two common normals such that \( d_2 = d_3 = 0 \). With these values, Eqs. (2.106) and (2.107) reduce to

\[ \begin{align*}
s_\alpha_4 s_\theta_5 &= \frac{h_2 - d_3 c_\alpha_4 - d_4}{a_3},
\end{align*} \]
\[
\begin{align*}
\text{(2.115)}: & \quad s a_4 \theta_2 = \frac{-n_z + c a_4 c a_5}{s a_5}, \\
\text{provided that } & \quad a_5 \neq 0 \text{ and } s a_5 \neq 0.
\end{align*}
\]

Again, Eqs. (2.114) and (2.115) contain only two unknown variables and hence are completely decoupled from Eqs. (2.108) and (2.109). We can eliminate \( \theta_5 \) by summing the squares of Eq. (2.114) and (2.115):

\[
\text{(2.116)}: \quad s^2 a_4 = \left( \frac{h_z - d_5 c a_4 - d_4}{a_5} \right)^2 + \left( \frac{-n_z + c a_4 c a_5}{s a_5} \right)^2.
\]

Equation (2.116) contains only one variable, \( \theta_1 \). We may convert it into a fourth-degree polynomial in \( t_1 \) by replacing \( s a_5 \) with \( 2t_1/(1 + t_1^2) \) and \( c a_5 \) with \( (1 - t_1^2)/(1 + t_1^2) \), where \( t_1 = \tan(\theta_1/2) \). Hence for each given end-effector position and orientation, there are at most four real solutions of \( \theta_1 \). Once \( \theta_1 \) is known, a unique solution of \( \theta_5 \) can be obtained by solving Eqs. (2.114) and (2.115) simultaneously for \( s \theta_5 \) and \( c \theta_5 \) and then applying the two-argument arctangent function.

When \( a_2 = a_3 = 0 \), the two scalar equations corresponding to the \( x \) and \( y \) components of Eq. (2.105) reduce to

\[
\begin{align*}
\text{(2.117)}: & \quad m_x c \theta_34 + m_y s \theta_34 = n_x c \theta_2 + n_y s \theta_2, \\
\text{(2.118)}: & \quad m_x s \theta_34 - m_y c \theta_34 = -n_x s \theta_2 + n_y c \theta_2.
\end{align*}
\]

Equations (2.117) and (2.118) contain two unknown variables, \( \theta_{34} \) and \( \theta_2 \). We may reduce these two equations to a single equation in one variable by the following procedure. Subtracting Eq. (2.122) \( \times s \theta_2 \) from (2.117) \( \times c \theta_2 \) yields

\[
\text{(2.119)}: \quad m_x c \theta_{234} + m_y s \theta_{234} = n_x.
\]

Adding Eq. (2.118) \( \times c \theta_2 \) to (2.117) \( \times s \theta_2 \) yields

\[
\text{(2.120)}: \quad m_x s \theta_{234} - m_y c \theta_{234} = n_y.
\]

Hence, corresponding to each solution set of \( \theta_1 \) and \( \theta_5 \), a unique solution of \( \theta_{234} \) can be obtained by solving Eqs. (2.119) and (2.120) for \( s \theta_{234} \) and \( c \theta_{234} \), and then applying the two-argument arctangent function.

Similarly, the two scalar equations corresponding to the \( x \) and \( y \) components of Eq. (2.104) reduce to

\[
\begin{align*}
\text{(2.121)}: & \quad g_x c \theta_{34} + g_y s \theta_{34} + a_3 c \theta_3 = h_x c \theta_2 + h_y s \theta_2 - a_2, \\
\text{(2.122)}: & \quad g_x s \theta_{34} - g_y c \theta_{34} + a_3 s \theta_3 = -h_x s \theta_2 + h_y c \theta_2.
\end{align*}
\]

\[
\text{Subtracting Eq. (2.122) \times s \theta_2 \times a_3 (k_3 c \theta_{23})}
\]

\[
\text{Adding Eq. (2.122) \times c \theta_2 to (2.117) \times s \theta_2 gives}
\]

\[
\text{Summing the squares of [Eq. (2.122)]}
\]

\[
\text{where}
\]

\[
\text{Hence, corresponding to each yields two solutions of } \theta_{23}. \text{ Once a unique solution of } \theta_2 \text{, we compute the solution sets.}
\]

\[
\text{2.7 METHOD OF SUCCESS}
\]

In this section we study a method of successive screw displacements. First, a screw displacement is derived. The successive screw displacements are then used to solve the position analysis of serial manipulators.

\[
\text{2.7.1 Transformation Based on Chasles' Theorem}
\]

Chasles' theorem states that the general relative displacement of a rigid body is a translation plus a rotation, regardless of how a rigid body is moved. Such a combination of translation and rotation is called a screw displacement (Bottema and Roth, 1979).
nly two unknown variables and (2.108) and (2.109). We can elimi-
nce for each given end-effector real solutions of \( \theta_1 \) is
ed by solving Eqs. (2.114) and then applying the two-argument
ns corresponding to the \( x \) and \( y \)
unknown variables, \( \theta_{23} \) and \( \theta_2 \).
\begin{align}
\theta_1 + n_x \theta_2, \\
\theta_2 + n_y \theta_2.
\end{align}

\begin{align}
\theta_1 + n_x \theta_2 = n_x, \\
\theta_2 + n_y \theta_2 = n_y.
\end{align}

\( \theta_1 \) and \( \theta_2 \), a unique solution of and (2.120) for \( \theta_{234} \) and \( \theta_{23} \),
\begin{align}
\theta_2 + h_y \theta_2 - a_2, \\
s_2 \theta_2 + h \theta_2.
\end{align}

\( \theta_2 \) and \( \theta_3 \), a unique solution of and (2.120) for \( \theta_{234} \) and \( \theta_{23} \),
\begin{align}
g_x \theta_{234} + g_y \theta_{234} + a_3 \theta_{23} &= h_x - a_2 \theta_2. \\
&= (2.123)
\end{align}

Adding Eq. (2.122) \times \theta_2 to (2.121) \times \theta_2 yields
\begin{align}
g_x \theta_{234} - g_y \theta_{234} + a_3 \theta_{23} &= h_y - a_2 \theta_2. \\
&= (2.124)
\end{align}

Summing the squares of [Eq. (2.123) - \( h_x \)] and [Eq. (2.124) - \( h_y \)] yields
\begin{align}
2a_3 (k_3 \theta_{23} + k_4 \theta_{23}) + k_5 = 0, \\
&= (2.125)
\end{align}

where
\begin{align}
k_3 &= g_x \theta_{234} + g_y \theta_{234} - h_x, \\
k_4 &= g_x \theta_{234} - g_y \theta_{234} - h_y, \\
k_5 &= -a_2^2 + a_3^2 + k_3^2 + k_4^2.
\end{align}

Hence, corresponding to each solution set of \( \theta_1, \theta_3, \theta_{234} \), Eq. (2.125) yields two solutions of \( \theta_{23} \). Once \( \theta_{23} \) is found, Eqs. (2.123) and (2.124) yield a unique solution of \( \theta_2 \). We conclude that there are at most eight possible solution sets.

2.7 METHOD OF SUCCESSIVE SCREW DISPLACEMENTS

In this section we study a method of analysis based on the concept of successive screw displacements. First, the transformation matrix associated with a screw displacement is derived. Then the concept of the resultant screw of two successive screw displacements is described. Then the concept is applied to the position analysis of serial manipulators.

2.7.1 Transformation Based on Screw Displacement

Chasles' theorem states that the general spatial displacement of a rigid body is a translation plus a rotation. A stronger form of the theorem states that regardless of how a rigid body is displaced from one location to another, the displacement can be regarded as a rotation about and a translation along some axis. Such a combination of translation and rotation is called a screw displacement (Bottema and Roth, 1979). In what follows we derive a homogeneous transformation based on the concept of screw displacement.
Figure 2.12 shows a point $P$ that is displaced from a first position $P_1$ to a second position $P_2$ by a rotation of $\theta$ about a screw axis followed by a translation of $t$ along the same axis. The rotation brings $P$ from $P_1$ to $P'_2$, and the translation brings $P$ from $P'_2$ to $P_2$. In the figure, $s = [s_x, s_y, s_z]^T$ denotes a unit vector along the direction of the screw axis, and $s_0 = [s_{ox}, s_{oy}, s_{oz}]^T$ denotes the position vector of a point lying on the screw axis. The rotation angle $\theta$ and the translational distance $t$ are called the screw parameters. The screw axis together with the screw parameters completely define the general displacement of a rigid body. Note that for a general displacement of a rigid body, the screw axis does not necessarily pass through the origin of the fixed frame.

The displacement equation due to a rotation about an axis passing through the origin was derived in Chapter 1. Hence we only need to take care of the fact that the screw axis does not pass through the origin and add the contribution due to a translation along the screw axis. Referring to Fig. 2.12, we observe that

$$r_1 = p_1 - s_0, \quad (2.126)$$
$$r_2 = p_2 - s_0 - ts. \quad (2.127)$$

Equation (2.128) can be written

$$A\mathbf{p} = \mathbf{p}_2 - \mathbf{s}_0 + ts + (\mathbf{p}_1 - \mathbf{s}_0)\cos \theta + \mathbf{s},$$

where the elements of the rotation the position of the origin, $A\mathbf{q}$, of th

$$q_x = ts_x - s_{ox}(a_1)$$
$$q_y = ts_y - s_{oy}(a_2)$$
$$q_z = ts_z - s_{oz}(a_3)$$

Substituting Eqs. (2.126) and (2.127),

$$\mathbf{p}_2 = s_0 + ts + (p_1 - s_0)\cos \theta + \mathbf{s}.$$
Substituting Eqs. (2.126) and (2.127) into (1.35), we obtain

$$
\mathbf{p}_2 = \mathbf{s}_o + t s + (\mathbf{p}_1 - \mathbf{s}_o)c\theta + s \times (\mathbf{p}_1 - \mathbf{s}_o)s\theta + [(\mathbf{p}_1 - \mathbf{s}_o)^T s] s (1 - c\theta).
$$

Equation (2.128) is known as Rodrigues's formula for the general spatial displacement of a rigid body. Expanding Eq. (2.128) and replacing \( \mathbf{p}_1 \) by \( ^B \mathbf{p} \) and \( \mathbf{p}_2 \) by \( ^A \mathbf{p} \), we obtain

$$
^A \mathbf{p} = ^A R_B ^B \mathbf{p} + ^A \mathbf{q},
$$

where the elements of the rotation matrix, \( a_{ij} \), are given by Eq. (1.37), and the position of the origin, \( ^A \mathbf{q} \), of the moving frame is given by

$$
\begin{align*}
q_x &= ts_x - s_{ox}(a_{11} - 1) - s_{oy} a_{12} - s_{oz} a_{13}, \\
q_y &= ts_y - s_{ox} a_{21} - s_{oy}(a_{22} - 1) - s_{oz} a_{23}, \\
q_z &= ts_z - s_{ox} a_{31} - s_{oy} a_{32} - s_{oz}(a_{33} - 1).
\end{align*}
$$

Equation (2.129) can be written as a homogeneous transformation:

$$
^A \hat{\mathbf{p}} = A \hat{\mathbf{p}}
$$

where \( A \) is a 4 x 4 transformation matrix the elements of which are given by

$$
\begin{align*}
a_{11} &= (s_{x}^2 - 1)(1 - c\theta) + 1, \\
a_{12} &= s_x s_y (1 - c\theta) - s_z s\theta, \\
a_{13} &= s_x s_z (1 - c\theta) + s_y s\theta, \\
a_{21} &= s_y s_x (1 - c\theta) + s_z s\theta, \\
a_{22} &= (s_{y}^2 - 1)(1 - c\theta) + 1, \\
a_{23} &= s_y s_z (1 - c\theta) - s_x s\theta, \\
a_{31} &= s_z s_x (1 - c\theta) - s_y s\theta, \\
a_{32} &= s_z s_y (1 - c\theta) + s_x s\theta, \\
a_{33} &= (s_{z}^2 - 1)(1 - c\theta) + 1, \\
a_{14} &= ts_x - s_{ox}(a_{11} - 1) - s_{oy} a_{12} - s_{oz} a_{13}, \\
a_{24} &= ts_y - s_{ox} a_{21} - s_{oy}(a_{22} - 1) - s_{oz} a_{23}, \\
a_{34} &= ts_z - s_{ox} a_{31} - s_{oy} a_{32} - s_{oz}(a_{33} - 1),
\end{align*}
$$

a spatial displacement.
The upper left $3 \times 3$ submatrix of $A$ represents the rotation of the rigid body. The upper right $3 \times 1$ submatrix represents the translation of the origin $Q$ (i.e., $a_{14} = q_x$, $a_{24} = q_y$, and $a_{34} = q_z$). This representation of a spatial displacement requires eight parameters: three associated with the direction of the screw axis, three associated with the location of the screw axis, one associated with the rotation angle, and one associated with the translational distance. However, only two of the three parameters associated with the direction of the screw axis are independent since they must satisfy the condition of a unit vector:

$$s^T s = 1.$$  \hfill (2.133)

Similarly, only two of the three parameters associated with the location of the screw axis are independent, since $s_o$ can be any point on the screw axis. For convenience, we may choose $s_o$ to be normal to the screw axis:

$$s_o^T s = 0.$$  \hfill (2.134)

Given the screw axis and screw parameters, we can compute the elements of the transformation matrix by Eq. (2.132). On the other hand, given the spatial displacement of a rigid body in terms of a rotation matrix, $^A R_B$, and a translation vector, $^A q$, we can compute the screw axis and the screw parameters as follows. The angle of rotation is given by

$$\theta = \cos^{-1} \frac{a_{11} + a_{22} + a_{33} - 1}{2}.$$  \hfill (2.135)

There are two solutions of $\theta$, one being the negative of the other. Once the rotation angle is known, the direction of the screw axis is computed by

$$s_x = \frac{a_{32} - a_{23}}{2s\theta},$$
$$s_y = \frac{a_{13} - a_{31}}{2s\theta},$$
$$s_z = \frac{a_{21} - a_{12}}{2s\theta}.$$  \hfill (2.136)

The translational distance is

and the screw axis location in Eq. (2.130) along $t$ there exists one solution corresponding to the $(s, s_o)$ screw axis process $+t$ translation along the $+$

2.7.2 Successive Screw Axis

We now apply the concept of successive open-loop chains. Figure 2.13 shows the fixed base by a dyad that is attached to the fixed base, and the screw axis located at $+t$ along the $+$

FIGURE 2.13. Two-link serial manipulator
successive screw displacements: that is, a rotation about the $n$th joint axis, followed by another about the $(n - 1)$th joint axis, and so on. Since all screw displacements take place about the joint axes at the reference position, the resulting screw displacement is obtained by premultiplying these screw displacements:

$$A_n = A_1A_2 \cdots A_{n-1}A_n.$$  

(2.139)

Using the method of successive screw displacements, only one fixed coordinate system and one end effector coordinate system are needed. The screw parameters used in Eq. (2.132) should not be confused with the Denavit-Hartenberg parameters. The joint variables of a screw displacement represent the actual angles of rotation and/or distances of translation needed to bring the end effector from a reference position to a target position. Specifically, for a revolute joint, $\theta_i$ is a variable and $t_i = 0$ identically, while for a prismatic joint, $t_i$ is a variable and $\theta_i = 0$ identically.

The D-H parameters do not represent the angle of rotation or the distance of translation about a joint axis. To obtain the actual displacements, it is necessary to subtract the joint variables associated with a reference position from that of a target position. One of the advantages of using successive screw displacements is that the reference position can be chosen arbitrarily. For example, it can be chosen at the home position of a robot, where all the information regarding the location of the end effector and the locations of the joint axes are known.

For direct kinematics, we compute Eq. (2.139) directly by using the given joint variables. For inverse kinematics, the left-hand side of Eq. (2.139) is given and the problem is to find the joint displacements needed to bring the hand to a desired location.

### 2.7.3 Position Analysis of an Elbow Manipulator

Figure 2.15 shows the schematic diagram of an elbow manipulator. In this manipulator, the second joint axis intersects the first perpendicularly, the third and fourth joint axes are parallel to the second, the fifth joint axis is perpendicular to the fourth with a small offset distance $a_4$, and the sixth joint axis intersects the fifth perpendicularly. We wish to solve the inverse kinematics problem of this manipulator using the method of successive screw displacements.

(a) Reference Position. First we identify a reference configuration with respect to which the displacement of the manipulator will be measured. Figure 2.16 shows such a reference configuration, where the first joint axis, $s_1$, points up vertically in the positive joint axes, $s_2$, $s_3$, and $s_4$, are all pointing in the positive $z$-direction, the hand coo, positive $x$-direction. The hand coo that the $w_0$-axis points in the positive $x$-direction.
rotation about the $n$th joint axis, and so on. Since all screw
axes at the reference position, the
premultiplying these screw dis-
placements, only one fixed coor-
dinates are needed. The screw
place be confused with the Denavit–
Hartenberg parameters. A screw displacement represents
of translation needed to bring the
target position. Specifically, for a
joint axes can be chosen arbitrarily. For ex-
ample, of a robot, where all the infor-
mation and the locations of the joint
were measured. Fig-
manipulator will be measured. Fig-
the first joint axis, $s_1$,
positive z-direction. At this reference position, the locations of the screw axes with respect to the fixed reference frame are listed in Table 2.5. The reference position of the end effector is

\[ u_3 = [0, 0, 1]^T, \quad v_0 = [0, -1, 0]^T, \quad w_0 = [1, 0, 0]^T, \quad \text{and} \quad p_0 = [a_2 + a_3 + a_4, 0, 0]^T. \]

(b) Target Position. Let the target position of the end effector be

\[ u = [u_x, u_y, u_z]^T, \quad v = [v_x, v_y, v_z]^T, \quad w = [w_x, w_y, w_z]^T, \quad \text{and} \quad p = [p_x, p_y, p_z]^T. \]

(c) Transformation Matrices. Substituting the coordinates of the joint axes into Eq. (2.132), we obtain the screw transformation matrices:

\[
A_1 = \begin{bmatrix}
    c\theta_1 & -s\theta_1 & 0 & 0 \\
    s\theta_1 & c\theta_1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}, \quad (A_1)^{-1} = \begin{bmatrix}
    c\theta_1 & s\theta_1 & 0 & 0 \\
    -s\theta_1 & c\theta_1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
    c\theta_2 & 0 & -s\theta_2 & 0 \\
    0 & 1 & 0 & 0 \\
    s\theta_2 & 0 & c\theta_2 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
A_3 = \begin{bmatrix}
    c\theta_3 & 0 & -s\theta_3 & a_2(1 - c\theta_2) \\
    0 & 1 & 0 & 0 \\
    s\theta_3 & 0 & c\theta_3 & -a_2s\theta_3 \\
    0 & 0 & 0 & 1
\end{bmatrix},
\]

(d) Inverse Kinematics. Given by

\[
A_1^{-1} = \begin{bmatrix}
    c\theta_4 & 0 & 0 & -a_1 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]
ions of the Elbow

\[
\begin{array}{lllll}
0, 0, 0 & 0, 0, 0
\end{array}
\]

The reference location of the end effector be

\[
\begin{array}{lllll}
1, 0, 0 & 1, 0, 0
\end{array}
\]

The transformation of the wrist center point \( P \) is given by

\[
p = A_1 A_2 A_3 A_4 p_0.
\]  

Multiplying both sides of the equation above by \( A_1^{-1} \), we obtain

\[
A_1^{-1} \begin{bmatrix}
p_x \\
p_y \\
p_z \\
1
\end{bmatrix} = A_2 A_3 A_4 \begin{bmatrix}
a_2 + a_3 + a_4 \\
0 \\
0 \\
1
\end{bmatrix}.
\]
Substituting $A_1^{-1}$ and Eq. (2.140) into (2.143) yields

$$p_x c \theta_1 + p_y s \theta_1 = a_2 c \theta_2 + a_3 c \theta_3 + a_4 c \theta_{234}, \quad (2.144)$$

$$-p_x s \theta_1 + p_y c \theta_1 = 0, \quad (2.145)$$

$$p_z = a_2 s \theta_2 + a_3 s \theta_3 + a_4 s \theta_{234}. \quad (2.146)$$

From Eq. (2.145), two solutions of $\theta_1$ are found immediately:

$$\theta_1 = \tan^{-1} \frac{p_y}{p_x}. \quad (2.147)$$

For this manipulator, the position and orientation are not decoupled. Therefore, we need to work on both simultaneously. Applying the transformation matrix to the approach vector $w$ gives

$$R_1^T w = R_2 R_3 R_4 R_5 w_0, \quad (2.148)$$

where $R_i$ denotes the upper left $3 \times 3$ submatrix of $A_i$. Expanding Eq. (2.148), we obtain

$$w_x c \theta_1 + w_y s \theta_1 = c \theta_{234} c \theta_5, \quad (2.149)$$

$$-w_x s \theta_1 + w_y c \theta_1 = s \theta_5, \quad (2.150)$$

$$w_z = s \theta_{234} c \theta_5. \quad (2.151)$$

Corresponding to each solution of $\theta_1$, Eq. (2.150) yields two solutions of $\theta_5$:

$$\theta_5 = \sin^{-1}(-w_x s \theta_1 + w_y c \theta_1). \quad (2.152)$$

That is, if $\theta_5 = \theta_5^*$ is a solution, $\theta_5 = \pi - \theta_5^*$ is also a solution. Once $\theta_1$ and $\theta_5$ are known, Eqs. (2.149) and (2.151) can be solved for $s \theta_{234}$ and $c \theta_{234}$. This leads to a unique solution for $\theta_{234}$:

$$\theta_{234} = \text{Atan2} \left[ \frac{w_z}{c \theta_5}, \frac{(w_x c \theta_1 + w_y s \theta_1) / c \theta_5} \right]. \quad (2.153)$$

Next, we solve Eqs. (2.144) and (2.146) for $\theta_2$ and $\theta_3$. For convenience, we rewrite Eqs. (2.144) and (2.146) as follows:

$$a_2 c \theta_2 + a_3 c \theta_3 = k_1, \quad (2.154)$$

$$a_2 s \theta_2 + a_3 s \theta_3 = k_2. \quad (2.155)$$

where $k_1 = p_x c \theta_1 + p_y s \theta_1$ is a solution. Once $\theta_3$ is known and (2.155) simultaneously for $\theta_3$, we can solve Eqs. (2.159) and (2.160) for $s \theta_6 = s \theta_5 (u_x c \theta_1 c \theta_{234} + u_y s \theta_1)$.

We conclude that there are a

2.7.4 Position Analysis

Figure 2.17 shows a 6-dof (Scheinman, 1969). In this manipulator (sliding pair) while all the others move by sliding up vertically in the positive third joint axis intersects the
where \( k_1 = p_x c \theta_1 + p_y s \theta_1 - a_4 c \theta_{234} \) and \( k_2 = p_x - a_4 s \theta_{234} \). Summing the squares of Eqs. (2.154) and (2.155), we obtain

\[
a_1^2 + a_2^2 + 2a_2a_3c \theta_1 = k_1^2 + k_2^2.
\]  

(2.156)

Hence

\[
\theta_3 = \cos^{-1} \frac{k_1^2 + k_2^2 - a_3^2 - a_4^2}{2a_2a_3}.
\]  

(2.157)

Therefore, corresponding to each solution set of \( \theta_1, \theta_2, \) and \( \theta_{234} \), there are at most two real solutions of \( \theta_3 \). Namely, if \( \theta_3^* \) is a solution, \( \theta_3 = -\theta_3^* \) is also a solution. Once \( \theta_3 \) is known, \( \theta_2 \) can be obtained by solving Eqs. (2.154) and (2.155) simultaneously for \( s \theta_2 \) and \( c \theta_2 \). This produces one solution of \( \theta_2 \). Finally, the solutions of \( \theta_4 \) are obtained from the relation \( \theta_4 = \theta_{234} - \theta_2 - \theta_3 \).

To solve for \( \theta_6 \), we apply the transformation to the unit vector \( u \):

\[
(R_1 R_2 R_3 R_4)^T u = R_5 R_6 u_0.
\]  

(2.158)

Expanding Eq. (2.158), we obtain

\[
\begin{align*}
-u_x c \theta_1 c \theta_{234} + u_x s \theta_1 c \theta_{234} + u_z s \theta_{234} &= s \theta_5 s \theta_6, \\
-u_x s \theta_1 + u_y c \theta_1 &= -c \theta_5 s \theta_6, \\
-u_x c \theta_1 s \theta_{234} - u_y s \theta_1 s \theta_{234} + u_z c \theta_{234} &= c \theta_6.
\end{align*}
\]  

(2.159) \hspace{1cm} (2.160) \hspace{1cm} (2.161)

We can solve Eqs. (2.159) and (2.160) for \( s \theta_6 \):

\[
s \theta_6 = s \theta_3 (u_x c \theta_1 c \theta_{234} + u_x s \theta_1 c \theta_{234} + u_z s \theta_{234}) - c \theta_3 (u_x s \theta_1 + u_y c \theta_1).
\]  

(2.162)

Equations (2.161) and (2.162) together determine a unique solution for \( \theta_6 \):

\[
\theta_6 = \text{Atan2}(s \theta_6, c \theta_6).
\]  

(2.163)

We conclude that there are at most eight real inverse kinematic solutions.

### 2.7.4 Position Analysis of the Stanford Arm

Figure 2.17 shows a 6-dof manipulator developed at Stanford University (Scheinman, 1969). In this manipulator, the third joint is a prismatic joint (or sliding pair) while all the others are revolute. The first joint axis, \( S_1 \), points up vertically in the positive z-direction. The second joint axis, \( S_2 \), intersects the first perpendicularly at point \( A \) and points in the positive x-direction. The third joint axis intersects the second perpendicularly at point \( B \) and points