Reading
Please read/review chapter 9 of Robotics, Vision and Control.

Questions

![Two DOF Robot manipulator](image)

1. Write the full equation of motion for the 2R arm above (i.e., $\tau_1$ and $\tau_2$ as a function of $\theta_1$ and $\theta_2$ and its derivatives)

Start with the masses of links: $m_1$ and $m_2$, to get the Mass Matrix recall (lecture 5)

$$M = \sum_{i=1}^{N} \left( m_i J_{\alpha_i}^T J_{\alpha_i} + J_{\alpha_i}^T I_C J_{\alpha_i} \right)$$

$$M = m_1 J_{\alpha_1}^T J_{\alpha_1} + J_{\alpha_1}^T I_{\alpha_1} J_{\alpha_1} + m_2 J_{\alpha_2}^T J_{\alpha_2} + J_{\alpha_2}^T I_{\alpha_2} J_{\alpha_2}$$

Note that:
- $m_i$ = the mass of the $i^{th}$ link
- $m_{ij}$ = the $ij$ element of the mass matrix
- $m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$

Note this is with respect to the configuration variable, not time.

On that subject, the derivative with respect to time would be:

$$\frac{d}{dt} m_{ij} = \sum_{k=1}^{N} m_{ijk} \dot{q}_k$$

The center of mass of each link is at the joint center, this $l_1 \equiv a_1/2$ and $l_2 \equiv a_2/2$

To compute the Jacobians ($J_v$ and $J_\omega$), we need to calculate the forward kinematics.
Recall that the position vectors (Lec 3, Slide 34) for a 2R arm are:

\[
0 \mathbf{P}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad \text{(this reads as “Position of Frame 1 as seen in 0”),} \quad 0 \mathbf{P}_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}
\]

Thus with respect to Frame \{0\}, the translational velocity Jacobians (i.e., the matrices that encode the differential relationship between joint velocities and workspace tip velocities) are found by direct differentiation of the position vectors \(0 \mathbf{P}_1\) and \(0 \mathbf{P}_2\).

\[
0 \mathbf{J}_v = \begin{bmatrix} -a_1 s_1 \\ a_1 c_1 \\ 0 \end{bmatrix}, \quad 0 \mathbf{J}_v = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} \\ a_2 c_{12} \end{bmatrix}
\]

\[
\mathbf{m}_i^0 \mathbf{J}_v^T \mathbf{J}_v = \begin{bmatrix} m_i a_i^2 \\ 0 \\ 0 \end{bmatrix}, \quad m_i^0 \mathbf{J}_v^T \mathbf{J}_v = \begin{bmatrix} m_1 \left( a_1^2 + a_2^2 + 2a_1a_2c_{12} \right) \\ m_2 \left( a_2^2 + a_1a_2c_{12} \right) \\ m_2 a_1^2 \end{bmatrix}
\]

The rotational velocity Jacobian matrices with respect to Frame \{0\} are given by

\[
\mathbf{J}_{\omega} = \begin{bmatrix} z_1 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{J}_{\omega} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}
\]

As both joints are revolute, these matrices are

\[
\mathbf{J}_{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{J}_{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}
\]

Thus, \(\mathbf{J}_{\omega} \mathbf{J}_{\omega}^T = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Finally, the mass matrix, \(\mathbf{M}\) is

\[
\mathbf{M} = \begin{bmatrix} m_1 a_1^2 + I_1 + m_2 \left( a_1^2 + a_2^2 + 2a_1a_2c_{12} \right) + I_2 \\ m_2 \left( a_2^2 + a_1a_2c_{12} \right) + I_2 \end{bmatrix} \begin{bmatrix} m_2 a_1^2 + I_2 \\ m_2 a_2^2 + I_2 \end{bmatrix}
\]

The Centrifugal and Coriolis Matrix \(\mathbf{v}\) is found directly by recalling Christoffel symbols (please review Christoffel symbols from dynamics and the mass notation from the previous page)

\[
b_{i,j,k} = \frac{1}{2} \left( m_{j,i} + m_{i,j} - m_{i,j} \right) \quad \text{and with} \quad b_{ii,i} = b_{i,i} = 0,
\]

the Centrifugal matrix becomes

\[
\mathbf{B} = \begin{bmatrix} 2b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} m_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \left( \frac{\partial \omega_1}{\partial \theta_1} \right) \\ 0 \end{bmatrix} = \begin{bmatrix} -m_2 a_1 a_2 s_{12} \\ 0 \end{bmatrix},
\]

and the Coriolis matrix can be written as

\[
\mathbf{C} = \begin{bmatrix} 0 & b_{22} \\ b_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{12} \\ -\frac{1}{2} m_{112} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \left( \frac{\partial \omega_1}{\partial \theta_2} \right) \\ 0 & \left( \frac{\partial \omega_2}{\partial \theta_2} \right) \end{bmatrix} = \begin{bmatrix} 0 & -m_2 a_1 a_2 s_{12} \\ m_2 a_1 a_2 s_{12} & 0 \end{bmatrix}
\]

Summing this together gives

\[
\mathbf{V} = \begin{bmatrix} -m_2 a_1 a_2 s_{12} \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & -m_2 a_1 a_2 s_{12} \\ m_2 a_1 a_2 s_{12} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix}
\]
The next factor to consider is gravity. While the problem does not specify a gravity direction, we assume it is acting parallel to the y-axis. This gives $\mathbf{g} = [0 \quad -g \quad 0]$. (Note that if we latter wish to assume that gravity is acting along the z-axis (into the page), this could be treated by setting $\mathbf{g} = [0 \quad 0 \quad -g]$)

With respect to Frame \{0\}, the gravity vector can be calculated as

$$ \mathbf{G} = -\left[ \mathbf{J}_{\mathbf{v}}^T \mathbf{m}_C \mathbf{g} + \mathbf{J}_{\mathbf{v}}^T \mathbf{m}_{C2} \mathbf{g} \right] $$

However, we have to be careful because the gravity acts at the mass center (which is represented by the notation C1 and C2). Again, recall that we have $l_1 = a_1/2$ and $l_2 = a_2/2$.

Given the structure of the problem, the Jacobians are be determined by inspection. Thus,

$$ \mathbf{G} = -\left[ -l_1 S_1 \quad l_1 C_1 \quad 0 \quad -m_1 g \right] - \left[ -a_1 S_1 - l_2 S_{12} \quad a_1 C_1 + l_2 C_{12} \quad 0 \quad -m_2 g \right] $n_2 S_{12} \quad l_2 C_{12} \quad 0 \quad 0

The Equations of Motion can be found by putting these terms together to give

(for review see also Lecture 4, Slide 30 and Lecture 5, Slide 7)

$$ \mathbf{\tau} = \mathbf{M} \left( \mathbf{\dot{\theta}} \right) \mathbf{\ddot{\mathbf{\theta}}} + \mathbf{v} \left( \mathbf{0}, \mathbf{\dot{\theta}} \right) + \mathbf{g} \left( \mathbf{0} \right) $$

$$ \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} m_1 a_1^2 + I_1 + m_2 \left( a_1^2 + a_2^2 + 2 a_1 a_2 C_2 \right) + I_2 \\ m_2 \left( a_2^2 + a_1 a_2 C_2 \right) + I_2 \end{bmatrix} + \begin{bmatrix} \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathbf{\ddot{\theta}}_1 \\ \mathbf{\ddot{\theta}}_2 \end{bmatrix}$$

$$ + \left( \mathbf{\dot{\theta}} \right) \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2 a_2 S_{12} \\ m_2 a_2 S_{12} \end{bmatrix} \begin{bmatrix} \mathbf{\dot{\theta}}_1^2 \\ \mathbf{\dot{\theta}}_2^2 \end{bmatrix} + \left( \mathbf{g} \right) \begin{bmatrix} \left( \frac{1}{2} m_1 + m_2 \right) a_1 C_1 + \frac{1}{2} m_2 a_2 C_{12} \\ \frac{1}{2} m_2 a_2 C_{12} \end{bmatrix} $$
2. Basic Motion Planning

Review the definition of a configuration space, workspace, and related terms.

After that, given the following start point, goal point, and configuration space obstacles, draw the full visibility graph and show the shortest path for a point robot.
Challenge Question:
Inverse Kinematics & Trajectory Generation

A small humanoid robot is being programmed to place a hat on its head. The objective is to place the hat in the position shown by the dashed outline in the figure below. Assume that the arm is composed of 3 revolute joints and is constrained to move in the plane of the page. The arm consists of 3 links with dimensions: $L_1=0.4$, $L_2=0.3$, $L_3=0.1$.

In order to place the hat on its head, assume that we must place the edge of the hat brim at a location 0.5m above its shoulder joint. The hat brim should be in a horizontal position and is gripped at its edge by the hand and is aligned with the last link of the arm. Please calculate/plot valid workspace (e.g., from the frame located at the right-most end of the brim where the robot is grasping it) and joint trajectories to place the hat correctly.