Introduction to State-Space

or

“States… in… spaaace!”

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Previously on METR4202...

• Robotics, kinematics, perception, oh my!

• You built robot arms to drive an end-effector to a specific point in space – some of you did well!

• …and many of you discovered that that’s hard to do accurately!
What you already know*

- Signals can be represented by transfer functions in the s-domain
- Roots of a transfer function’s denominator (poles) indicate the stability of the system
- Poles move around under feedback control
  - Feedback can stabilise an unstable system

*If you have no idea what I’m talking about, now is the time to mention it.
A quick recap

- Differential equations are used to represent the dynamics of systems in time:
  \[ \dot{x} = f(x, t) \]

- For linear systems, we use the Laplace transform to represent differential operators:
  \[ sx = \mathcal{L}\{f(x, t)\} \]
A quick recap

- For SISO LTI* systems, output $y$ is a linear function of input $u$ in the Laplace domain:
  
  $$y = Hu$$

  $H$ is the ‘transfer function’ relating $y$ and $u$

- We use block diagrams to represent such systems in convenient graphical form:

*Single Input Single Output, Linear Time Invariant
State-space lolwut?

• A ‘clean’ way of representing systems

• Easy implementation in matrix algebra

• Simplifies understanding Multi-Input-Multi-Output (MIMO) systems
Introduction to states*

- Introductory brain-teaser:
  - If you have a step response model of a system with integration, how do you represent non-zero initial conditions?

Eg. how would you setup a simulation of a step response, mid-step?

*Not-insubstantial portions of these slides are based on Franklin, Powel and Enami-Naeni
Store the values

- The time-history of dynamic systems can be encapsulated by ‘states’.
- A state is any previous value upon which future outputs depend:
  - Eg. velocities, altitude, displacement, charge potential, stored magnetic fields, etc.

All the state values of a system are stored in a single column vector $x$, which is collectively termed “the system’s state”.
Finding states

- Linear systems can be written as networks of simple dynamic elements:

\[ H = \frac{s + 2}{s^2 + 7s + 12} = \frac{2}{s + 4} + \frac{-1}{s + 3} \]
Finding states

- We can identify the nodes in the system
  - These nodes contain the integrated time-history values of the system response
  - We call them “states”
### Linear system equations

- We can represent the dynamic relationship between the states with a linear system:

\[
\begin{align*}
\dot{x}_1 &= -7x_1 - 12x_2 + u \\
\dot{x}_2 &= x_1 + 0x_2 + 0u \\
y &= x_1 + 2x_2 + 0u
\end{align*}
\]
State-space representation

- We can write linear systems in matrix form:

\[
\dot{x} = \begin{bmatrix} -7 & 12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 1 & 2 \end{bmatrix} x + 0u
\]

Or, more generally:

\[
\dot{x} = Ax + Bu
\]

\[
y = Cx + Du
\]

“State-space equations”
Why “State-space”? 

- State vector entries can be thought of as coordinates in a space; hence ‘state-space’

\[ \mathbb{R}^3 \]
State-space representation

• State-space matrices are not necessarily unique representations of a dynamic system
  – There are several common forms (here’s two)

• Control canonical form
  – Each node – each entry in $x$ – represents a state of the system (each order of $s$ maps to a state)

• Modal form
  – Diagonals of the state matrix $A$ are the poles ("modes") of the transfer function
Other forms

• There are other representations are useful for other purposes:
  – Observer canonical form
  – Phase variable canonical form
  – Jordan canonical form
  – Etc.

But you don’t need to know about those for this course, but check out
http://www.ece.rutgers.edu/~gajic/psfiles/canonicalforms.pdf
if you’re interested!

For now, let’s focus on CCF and MF
Control canonical form

- CCF matrix representations have the following structure:

\[
\begin{bmatrix}
-a_1 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\
1 & 0 & & 0 & 0 & 0 \\
0 & 1 & & & & \\
\vdots & & \ddots & & & \\
0 & & & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 \\
\end{bmatrix}
\]

Pretty diagonal!
Modal form

- MF matrix representations have the following structure:

\[
\begin{bmatrix}
-p_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -p_2 & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & -p_{n-2} & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & -p_{n-1} & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & -p_n & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -p_n
\end{bmatrix}
\]

Also pretty diagonal!
EXAMPLE TIME

STOP
BREAK TIME

GIVE WAY
Cool robotics share

The McDonnell Douglas DC-X “Delta Clipper” was a demonstrator developed to explore vertical rocket landing and reusable single-stage to orbit technology. The DC-X used nonlinear state-space control in its flight attitude regulation avionics. Testing of the prototype was carried out by NASA, but the design was a competitor for NASA’s own X-33 craft. Despite a rigorous testing schedule, the public success of the DC-X (compared to the embarrassing delays plaguing the X-33) led to its continued development. However, punishing deadlines and burned-out ground crew eventually resulted in a landing accident that severely damaged the craft – the program was quickly cancelled.
DC-X Delta Clipper

DC-X Flight #8
July 7, 1995
The "Swan Dive" Test
(view from ground)
State variable transformation

• Important note!
  – The states of a control canonical form system are not the same as the modal states
  – They represent the same dynamics, and give the same output, but the vector values are different!

• However we can convert between them:
  – Consider state representations, \( x \) and \( q \) where
    \[
    x = Tq
    \]

    \( T \) is a “transformation matrix”
State variable transformation

• Two homologous representations:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

and

\[
\begin{align*}
\dot{q} &= Fq + Gu \\
y &= Hq + Ju
\end{align*}
\]

We can write:

\[
\begin{align*}
\dot{x} &= T\dot{q} = ATq + Bu \\
\dot{q} &= T^{-1}ATq + T^{-1}Bu
\end{align*}
\]

Therefore, \( F = T^{-1}AT \) and \( G = TB \)

Similarly, \( C = HT \) and \( D = J \)
Consider…

• What if we try to turn an arbitrary state description into control canonical form?
  
  – We expect that for $AT^{-1} = T^{-1}F$:
  
  $\begin{bmatrix}
-a_1 & -a_2 & \ldots & -a_n \\
1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & 0 & 1 & 0
\end{bmatrix}\begin{bmatrix}
t_1 \\
t_2 \\
\vdots \\
t_n
\end{bmatrix} = \begin{bmatrix}
t_1F \\
t_2F \\
\vdots \\
t_nF
\end{bmatrix}$

  where $t_i$ are the rows of $T^{-1}$

  Then, $t_2 = t_3F$, and $t_1 = t_2F = t_3F^2$, etc…

  Note: $t_n$ is a row and $t_nF$ yields a row
Consider…

- $T^{-1}G = B$, in control canonical form yields

$$
\begin{bmatrix}
t_1G \\
\vdots \\
t_nG
\end{bmatrix}
= \begin{bmatrix} 1 \\
0 \\
\vdots \\
0
\end{bmatrix}
$$

The two results together give:

$t_nG = 0$

...  

$t_2G = t_n F^{n-1}G = 0$

$t_1G = t_n F^nG = 1$
Consider...

• Look at the last entry for $t_3$...
  – We can write this as:
    \[
    t_3 [G \ FG \ \ldots \ F^n G] = [0 \ 0 \ \ldots \ 1]
    \]

Or

\[
 t_3 = [0 \ 0 \ \ldots \ 1] \mathcal{C}^{-1}
\]

where \( \mathcal{C} = [G \ FG \ F^2 G \ \ldots \ F^{n-1} G] \)

This is called the “controllability matrix”
Controllability matrix

• To convert an arbitrary state representation in $F$, $G$, $H$ and $J$ to control canonical form $A$, $B$, $C$ and $D$, the controllability matrix

$$\mathcal{C} = [G \quad FG \quad F^2G \quad \ldots \quad F^{n-1}G]$$

must be invertible (i.e. full rank).

>deep think<

Why is it called the “controllability” matrix?
Controllability matrix

• If you can write it in CCF, then the system equations must be linearly independent.

• Transformation by any invertible matrix preserves the controllability of the system.

• Thus, a invertible controllability matrix means $x$ can be driven to any value.
Kind of awesome

• The controllability of a system depends on the particular set of states you chose

• You can’t tell just from a transfer function whether all the states of \( x \) are controllable

• System poles are the Eigenvalues of \( F \), \( (p_i) \)
State evolution

- Consider the system matrix relation:
  \[ \dot{x} = Fx + Gu \]
  \[ y = Hx + Ju \]

  The time solution of this system is:
  \[ x(t) = e^{F(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)} Gu(\tau)d\tau \]

  If you didn’t know, the matrix exponential is:
  \[ e^{Kt} = I + Kt + \frac{1}{2!} K^2 t^2 + \frac{1}{3!} K^3 t^3 + \cdots \]
Stability

- We can solve for the natural response to initial conditions $x_0$:
  \[ x(t) = e^{p_i t} x_0 \]
  \[ \therefore \dot{x}(t) = p_i e^{p_i t} x_0 = F e^{p_i t} x_0 \]

Clearly, a system will be stable provided $\text{eig}(F) < 0$
Characteristic polynomial

- From this, we can see $F x_0 = p_i x_0$
  
  or, $(p_i I - F)x_0 = 0$

Which is true only when $\det(p_i I - F)x_0 = 0$

Aka. the characteristic equation!

- We can reconstruct the CP in $s$ by writing:
  
  $\det(sI - F)x_0 = 0$
Great, so how about control?

- Now that we have a state space model, how do we make the system stable, or converge to desired states?

  Easy: Feedback!

- Given $\dot{x} = Fx + Gu$, if we know $F$ and $G$, we can design a controller $u = -Kx$ such that
  
  $\text{eig}(F - GK) < 0$
Full state feedback

• If we have full measurement and control of the states of \( x \), we can position poles of the closed-loop system in arbitrary locations.
  – Modal form makes this straightforward:

\[
F - GK = \begin{bmatrix}
-p_1 - G \cdot K_{1j} \\
-p_2 - G \cdot K_{2j} \\
-p_3 - G \cdot K_{3j}
\end{bmatrix}
\]

Of course, this hardly ever happens in reality.

Brain teaser: Why?
Example: PID control

• Consider a system parameterised by three states: $x_1, x_2, x_3$ where $x_2 = \dot{x}_1$ and $x_3 = \dot{x}_2$

$$\dot{x} = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix} x - Ku$$

$$y = [0 \ 1 \ 0] x + 0u$$

$x_2$ is the output state of the system; $x_1$ is the value of the integral; $x_3$ is the velocity.
Example: PID control

- We can choose $K$ to move the eigenvalues of the system as desired:

\[
\begin{vmatrix}
1 - K_1 & 1 - K_2 \\
-2 - K_3 & 0
\end{vmatrix} = 0
\]

All of these eigenvalues must be positive.

It’s straightforward to see how adding derivative gain $K_3$ can stabilise the system.
In reality…

• You can never measure or apply control action to all states directly.
  – The majority of system states will be hidden to the control engineer.

But we can pretend!

• We can design a controller as if we did, using an estimate – an educated guess.
Observers

- Observers (aka “estimators”) are used to infer the hidden states of a system from measured outputs.

A controller is designed using estimates in lieu of full measurements.
Observers

• The state estimate can be treated like a control system itself
  – Dynamics to update the estimate:
    \[ \dot{\hat{x}} = F\hat{x} + Gu \]
  – By measuring an ‘error signal’ from the difference between the real output measurement and the output estimate, \( \tilde{x} = x - \hat{x} \), the state estimate can be shown to converge
Observers

- Just like you might expect:
  \[
  \dot{x} = F\dot{x} + Gu + L(y - H\dot{x})
  \]
  \[
  \therefore \dot{x} = (F - LH)\ddot{x}
  \]

Choose \(L\) to make \(\ddot{x}\) converge to 0
Observability

• The ability to infer these values is called “Observability”
  – This is the dual of controllability; a system that is observable is also controllable and vice versa.
  – Observability matrix:

\[
\mathcal{O} = \begin{bmatrix}
    H \\
    HF \\
    \vdots \\
    HF^{n-1}
\end{bmatrix}
\]
Just scratching the surface

- There is a lot of stuff to state-space control

- One lecture (or even two) can’t possibly cover it all in depth

  Go play with Matlab and check it out!
EXAMPLE TIME

STOP
Quick plug*

“Feedback Control of Dynamic Systems” by Franklin, Powell and Emami-Naeini.

“Control System Design: An Introduction to State-Space Methods” by Friedland

* No, they’re not paying me – they’re just really good books!
Tune-in next time for...

Control/Planning Under Uncertainty
Starring
Hanna Kurniawati!

Fun fact: Saying “eigenvalue” makes you feel smarter!
And now a word from our hideous sponsor!
EXAM TIPS

Double-check the examination timetable to ensure you have the correct date, time and venue for your examination.

I CAN’T SIT AN EXAM – WHAT SHOULD I DO?
If you can’t sit because of medical reasons or other unavoidable circumstances, you may be eligible to apply for a deferred exam. Beware: misreading the exam timetable is NOT an approved reason for applying for a deferred exam.
Check out myAdvisor www.uq.edu.au/myadvisor/unable-to-sit-an-examination

WHAT HAPPENS IF I’M LATE?
If you are:
A bit late ... during perusal (reading) time, you take the exam with your classmates using the remaining perusal and working time.
A bit later ... after perusal time but within the first 30 minutes of working time, you take the exam in the timetabled room using the working time remaining.
A bit later still ... after the first 30 minutes of working time you will not be permitted to undertake the examination. You will not be entitled to sit a deferred examination unless you meet the requirements of the University’s General Award Rules www.uq.edu.au/student/GeneralRules2012/2012GARs.pdf

MORE QUESTIONS?
Look at the examination information at www.uq.edu.au/myadvisor/examinations or email examinations@uq.edu.au
EXAM TIPS

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WHAT CAN I TAKE INTO THE EXAM?
- Yourself
- Student ID card
- Writing implements and any authorised material

GOT A MOBILE PHONE?
Turn it off and it goes under your chair in the exam room. Students found with mobile phones on their person will be dealt with under the University’s misconduct provisions.

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WHAT SHOULDN’T I BRING TO AN EXAM?
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- Any unauthorised stuff! (your lecturer will inform you whether you are allowed to bring any written or electronic aids).

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A “University approved calculator” label is needed for all calculators (other than Casio FX82 series) for use in examinations. These labels are available from the Student Centre. Check out myAdvisor www.uq.edu.au/myadvisor/exam-calculators

WHAT HAPPENS IF I FORGET TO TAKE ID TO AN EXAM?
If you don’t have your student ID card, you will be refused entry to the examination room and will be directed to the Student Centre for further advice.
Discretisation FTW!

- We can use the time-domain representation to produce difference equations!

$$x(kT + T) = e^{FT} x(kT) + \int_{kT}^{kT+T} e^{F(kT+T-\tau)} Gu(\tau) d\tau$$

Notice $u(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series. We can structure this by using the form:

$$u(\tau) = u(kT), \quad kT \leq \tau \leq kT + T$$
Discretisation FTW!

• Put this in the form of a new variable:
  \[ \eta = kT + T - \tau \]

Then:
\[
\begin{align*}
  \mathbf{x}(kT + T) &= e^{FT} \mathbf{x}(kT) + \left( \int_{kT}^{kT+T} e^{F\eta} d\eta \right) \mathbf{G}\mathbf{u}(kT) \\
  \text{Let’s rename } &\Phi = e^{FT} \text{ and } \Gamma = \left( \int_{kT}^{kT+T} e^{F\eta} d\eta \right) \mathbf{G}
\end{align*}
\]
Discrete state matrices

So,

\[ x(k + 1) = \Phi x(k) + \Gamma u(k) \]
\[ y(k) = Hx(k) + Ju(k) \]

Again, \( x(k + 1) \) is shorthand for \( x(kT + T) \)

Note that we can also write \( \Phi \) as:

\[ \Phi = I + FT\Psi \]

where

\[ \Psi = I + \frac{FT}{2!} + \frac{F^2T^2}{3!} + \cdots \]
Simplifying calculation

- We can also use $\Psi$ to calculate $\Gamma$
  - Note that:
    \[
    \Gamma = \sum_{k=0}^{\infty} \frac{F^k T^k}{(k + 1)!} T G
    \]
    \[
    = \Psi T G
    \]

$\Psi$ itself can be evaluated with the series:
\[
\Psi \approx I + \frac{FT}{2} \left( I + \frac{FT}{3} \left[ I + \cdots \frac{FT}{n-1} \left( I + \frac{FT}{n} \right) \right] \right) \]
We can apply the $z$-transform to our system:

$$(zI - \Phi)X(z) = \Gamma U(k)$$

$$Y(z) = HX(z)$$

which yields the transfer function:

$$\frac{Y(z)}{X(z)} = G(z) = H(zI - \Phi)^{-1} \Gamma$$
State-space control design

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:
    \[ u = -Kx \]
    such that \( \det(zI - \Phi + \Gamma K) = \alpha_c(z) \)
    where \( \alpha_c(z) \) is the desired control characteristic equation

Predictably, this requires the system controllability matrix
\[ \mathcal{C} = [\Gamma \quad \Phi \Gamma \quad \Phi^2 \Gamma \cdots \quad \Phi^{n-1} \Gamma] \] to be full-rank.