Q1. Exploring the Deep Blue ZOH [15 %]

The Zero Order Hold (ZOH) provides a mathematical model of a sample and data hold operation (often for given sampling period $T$). Such operations are common when there is a continuous plant, $G(s)$, that has to interface with a discrete time controller and sensor system (with a DAC+hold and a sample+ADC) as shown below.

For such systems we showed that an equivalent model may be found via

$$G(z) = (1 - z^{-1})Z\left(\frac{G(s)}{s}\right)$$

(a) Suppose $G_a(s) = \frac{a}{s^4}$, what is $G_a(z)$?
(b) Suppose $G_b(s) = \frac{a^2+b^2}{(s^2+a^2)^2}$, what is $G_b(z)$?

(c) The derivation of the above equivalency assumed that the DAC and the ADC are synchronized, hence the one “hold” delay and the $(1 - z^{-1})$ factor. Imagine, instead, that they are not in synchrony. As such there is now a fixed latency of $\tau$ seconds. What would $G^*(z, \tau)$ be so as to get the equivalence under this condition?

(d) Using the result in part (c), what is $G^*(z, T)$ for the systems $G_a(s)$ and $G_b(s)$ given in parts (a) and (b) respectively.

Hints:
- For part (b), consider $Z\left(\frac{G_b}{s}\right)$ first
- For part (d), how does the result in (c) simplify when $\tau=T$? where $T$ is the sampling period.
This question explores discrete time equivalents for a given sampling period $T$.

Given the following plants (Plants A, B and C respectively):

$$G_a(s) = \frac{10s+10}{s^2 + 10s + 10}$$
$$G_b(s) = \frac{10s+10}{s^2 + 100}$$
$$G_c(s) = \frac{10s+10}{s^2 + 1000}$$

(a) Determine their equivalents in the Z domain (i.e. $G_a(z)$, $G_b(z)$ and $G_c(z)$) using the Trapezoid Rule.

(b) Determine their equivalents in the Z domain (i.e. $G_a^{ZOH}(z)$, $G_b^{ZOH}(z)$ and $G_c^{ZOH}(z)$) if they were in a sample and data hold operation (i.e., using a ZOH model).

(c) Plot the open-loop step response of the ZOH equivalents (i.e. $G_a^{ZOH}(z)$, $G_b^{ZOH}(z)$ and $G_c^{ZOH}(z)$) for $T=1$ ms

**note:** please overlay these responses on the same graph.

(d) For Plant C, determine the Trapezoidal and ZOH equivalents for $T=10$ ms. Then please plot (as an overlay) the open-loop step responses for the (i) continuous [$G_c(s)$], (ii) Trapezoidal [$G_c(z)$] and (iii) ZOH [$G_c^{ZOH}(z)$] cases. **Please then clearly explain the result.**

**note:** consider how the responses vary and if this is as expected

(e) For the ZOH equivalent of Plant C ($G_c^{ZOH}(z)$), plot (as an overlay) the open-loop step responses for $T=[1$ ms, $10$ ms, $100$ ms and $1$ second]. **Please then clearly explain the result.**

Hints:

- For all the graphs, please remember to label the axes clearly. When plotting overlays please be sure to select markers, line types and line colours clearly and professionally as well as to include a legend.
- For part (c, d and e), it might be helpful to write a program (in any language of your choosing). In Matlab, the following commands might be helpful: `tf`, `c2d` and `step`.
- In part (e), it might help to determine and consider the characteristic equation and natural frequency ($\omega_0$) of the plant.
Q3. Closed Loop System Ahoy!  
(Now with digital computer compensation)

We consider first-order digital compensation with a second-order plant. To start, imagine we have a plant given by

\[ G(s) = \frac{1}{(s+1)(s+10)} \]

Professor Ahab wants to help (but is only comfortable with controllers in the s-domain) and suggests:

\[ C(s) = \frac{300s}{s+42} \]

We recall that digital compensator may be thought of a continuous equivalent, given by:

\[ D(z) = \frac{K(s-A)}{z-B} = Z\left\{\frac{K(s+a)}{s+b}\right\} \]

Where \( A = e^{-aT} \) and \( B = e^{-bT} \) (and \( K \) is distinct from \( k \)):

(a) Using \( D(z) \) above, what is an expression for \( K \) when \( s = 0 \) (i.e. the DC gain)?
(b) Using the controller designed by Prof Ahab (the result in (a)), what are \( K, A \) and \( B \) such that we have unity gain at DC?
(c) Determine the closed-loop (negative) unity feedback continuous-time transfer function, \( H(s) \), for the controller and plant together.
(d) Determine the discrete-time transfer function, \( H(z) \), for the closed-loop (negative) unity feedback case in (c) assuming 100 Hz, ZOH sampling.
(e) Please plot the step response for the case in (c) and (d) (i.e. \( H(s) \) and \( H(z) \))

(f) As they say, timing is everything. Now consider the case of a small, 1 sample delay (or in this case 10 ms) for the solutions found in part (d). Please determine the new, delayed closed-loop (negative) unity feedback transfer function, \( H'(z) \) and then plot its step response. Please then clearly explain the result.

Hints:
- For part (a), what is the simplification of \( D(z) \) when \( s = 0 \)? What is \( Z\{s=0\} \)?
- For part (c), what is the loop gain? Depending on the approach one takes, it might help to compute \( C(s)G(s) \) first.
- For parts (d), (e) and (f), it might be helpful to write a program (in any language of your choosing). In Matlab, the following commands might be helpful: t.f, c2d, ss, and step.
Dr Nemo, an associate of Professor Ahab, has been engineering a new underwater robot, the Nautilus. The state-space representation of this system is:

\[
A = \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ -12 \\ 5 \\ 2 \end{bmatrix}, \quad C = [3 \ 0 \ 0 \ 4], \quad D = [0]
\]

(a) What is the state-transition matrix \( \Phi(s) \) for this system? 
[Note: Recall that \( \Phi(s) = [sI - A]^{-1} \)]

(b) What is the transfer function for this system? 
[Note: Recall that \( H(s) = C\Phi(s)B + D \)]

(c) Is this system controllable? 
If so, provide a Proportional control law that can serve as a stable regulator
If not, please advise how many of the modes are controllable? 

[Note: Recall that the Controllability Test Matrix Q for this case would be \( Q = [B \ AB \ A^2B \ A^3B \ ] \)]

Some Optional Remarks:

Why is \( \Phi(s) = [sI - A]^{-1} \)? This comes from the solution to the state space equations \( x = Ax + Bu \), \( y = Cx + Du \) is \( X(s) = (sI - A)^{-1}[x(0) + BU(s)] \) and we define \( \Phi(s) \) from this. Conversely, the output is \( Y(s) = C\Phi(s)x(0) + [C\Phi(s)B + D]U(s) \). For a case where there is zero-input response, this can be simplified and written as a transfer function as \( \frac{Y(s)}{U(s)} = H(s) = C\Phi(s)B + D \) (note that this notation is not universal, some define \( \frac{Y(s)}{U(s)} = G(s) \)). Finally from this we can also obtain a system characteristic polynomial via \( |\Phi| \) or \( |sI - A| \).

Similarly, in the Z domain we will get \( [zI - \Phi]X(z) = \Gamma U(z) \), and \( Y(z) = HX(z) \). Hence, \( \frac{Y(z)}{U(z)} = H[zI - \Phi]^{-1}\Gamma \)

⇒ [ Errata (24/5): B updated (now B(4)=2 so as to help simplify this, if you did this with the old value state your assumptions and submit the solution) ]
Q5. Charging Ahead With State-Space Models!  

The Nautilus is an electric vessel. The drive control circuit may be modeled as a controllable RLC circuit (shown below) driven a varying current source, \( u(t) \). In this problem, we seek to analyse and control this circuit.

\[
\begin{align*}
\text{L} & \quad \text{i}_L \\
\text{C} & \quad v_C \\
\text{R} & \quad v_G \\
\end{align*}
\]

(a) Assume a state vector \( x = [x_1, x_2] = [v_C, i_L] \), a control input current \( u \) and an output voltage \( y \) as measured across the resistor. Please describe the operation of the circuit as first order differential equations of time in terms of the state variables \( x_1 \) and \( x_2 \).

(b) Now, kindly determine a state-space model for this circuit.

(c) Using the previous results, please determine \( \Phi(s) \) for this system.

(d) Now assume \( R=10 \, \text{k}\Omega \), \( L=1 \, \text{H} \) and \( C=0.01 \, \text{mF} \). Please determine \( \Phi(t) \) for this system for a time slice of 0.1 seconds.

(e) Using the previous results, and assuming \( u(t) \) is a delayed unit step current (at \( t=1 \) second) and assuming a initial values (\( t=0 \)) of \( x_1(0) = 3.3 \) volts, \( x_2(0) = 0.1 \) amp, \( u(0) = 0 \) amps, please determine the value of the system (\( x \)) at \( t = 1 \) second and plot the value each state variable (\( [x_1, x_2] \)) as a function of time for \( t = [0...10] \) seconds.

(f) In the above analysis (case (e)), what would happen if \( u(t) \) was a s 50Hz sine wave of 1 amp instead?

(g) State-space models can also help find (or be transformed to) transfer functions. Using the results from above find \( G(s) = \frac{Y(s)}{U(s)} \). Is this function as you expected? **Explain.**

(h) State-space models can also help find discrete-time representations easily (assuming a ZOH). Using the previous results, please find a discrete-time representation for the state (\( x[k+1] \)) and the output (\( y[k] \)). Is the result as you expected? **Explain.**

**Reading Guide:**
Before solving the question, please review the following material:
- Lathi §13.3 (pp. 798-811)
- Franklin, Powell and Workman, §6.2 (pp. 239-250)
- Matlab’s [expm](https://www.mathworks.com/help/matlab/ref/expm.html) and [lsim](https://www.mathworks.com/help/matlab/ref/lsim.html) documentation might be helpful.
Notes:

- Part (f): for a SISO LTI system, the transfer function may be found via $G(s) = C\Phi(s)B + D$
- Part (g): For a SISO LTI system with control applied via a ZOH, we have
  
  $$x(k + 1) = \Phi x(k) + \Gamma u(k)$$
  $$y(k) = C x(k)$$

  ○ Where $\Phi = e^{A T}$ and $\Gamma = \int_0^T e^{A \eta} dB$

Hints:

- For part (a), we are asking you to determine $\frac{dx_1}{dt}$, $\frac{dx_2}{dt}$ and $y(t)$?
- For part (b), we are asking you to determine the $A$, $B$, $C$ and $D$ matrices as a general function of the R, L, and C components of the circuit
- For part (c), recall that $\Phi(s) = (sI - A)^{-1}$
- For part (d), recall that $\Phi(t) = e^{At}$, where $e^{At}$ is the matrix exponential
- Recall that the matrix exponential may be calculated by:
  
  $$e^{At} = \exp(At) = I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^kt^k}{k!} + \ldots$$

- Recall that the $\Gamma$ integral may be calculated by:
  
  $$\Gamma = \sum_{k=0}^{\infty} \frac{A^kT}{(k+1)!} B = (I + AT + \frac{A^2T^2}{2} + \ldots + \frac{A^kT^k}{k!} + \ldots)B$$

- For additional practice, it might help at this point to determine the unforced system solution (i.e. $u(t) = 0$) for initial conditions of $x_1(0) = \overline{x_1}$ and $x_2(0) = \overline{x_2}$
- For part (e) it might (or might not) help to consider the $(\dot{x} = Ax + Bu, \ y = Cx)$ for the given values.
- For part (g), reviewing FPW §6.2 (particularly p. 239) might be helpful.

Last Word:

Finally -- for this is the last ELEC3004 Problem Set Question -- a little philosophy...

"If you want to build a ship, don’t drum up the men to gather wood, divide the work and give orders. Instead, teach them to yearn for the vast and endless sea."

—Antoine de Saint-Exupery, The Wisdom of the Sands

May the wonder always be there. Thank you very much! ☺️