Discrete Time Analysis & Z-Transforms

ELEC 3004: **Systems**: Signals & Controls
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Lecture 7

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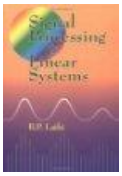
Lecture Schedule:

Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Systems: Linear Differential Systems
3	14-Mar	Sampling Theory & Data Acquisition
	16-Mar	Aliasing & Antialiasing
4	21-Mar	Discrete Time Analysis & Z-Transform
	23-Mar	Second Order LTID (& Convolution Review)
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
5	4-Apr	Digital Filters (IIR)
	6-Apr	Digital Windows
6	11-Apr	Digital Filter (FIR)
	13-Apr	FFT
	18-Apr	Holiday
	20-Apr	
	25-Apr	
7	27-Apr	Active Filters & Estimation
8	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	11-May	Digitial Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response
	25-May	Applications in Industry
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

Today

- Chapter 8 (**Discrete-Time Signals and Systems**)
  - § 8.1 Introduction
  - § 8.2 **Some Useful Discrete-Time Signal Models**
  - § 8.3 **Sampling Continuous-Time Sinusoids & Aliasing**
  - § 8.4 Useful Signal Operations
  - § 8.5 Examples of Discrete-Time Systems

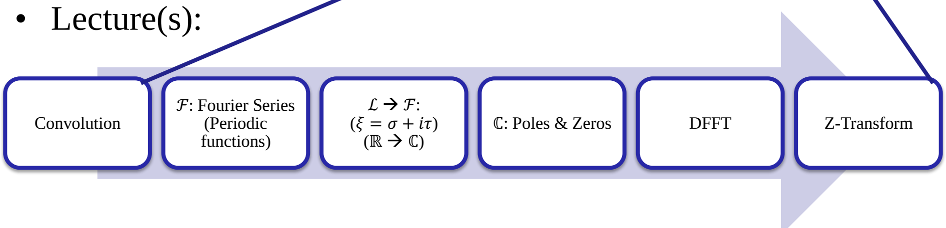
- Chapter 11 (**Discrete-Time System Analysis Using the *z*-Transform**)
  - § 11.1 The $\mathcal{Z}$-Transform
  - § 11.2 Some Properties of the *Z*-Transform

Next Time

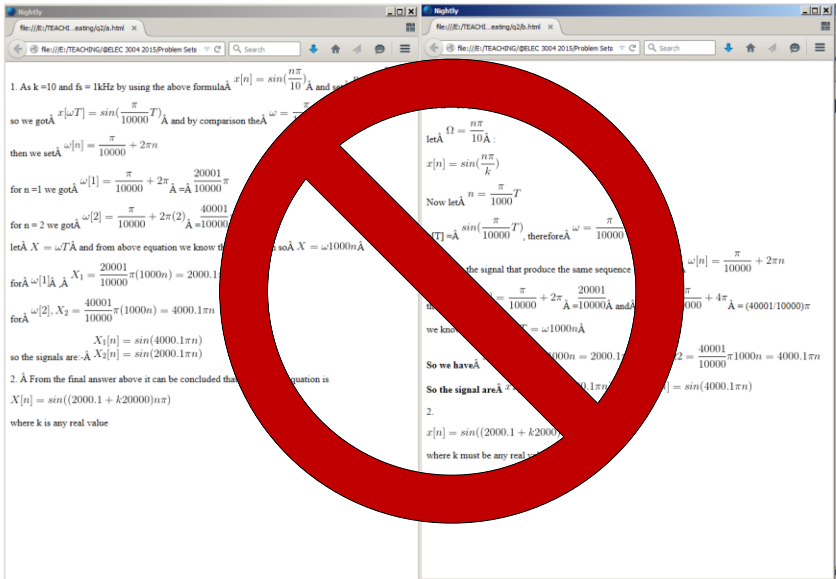
Lecture Overview

- Course So Far:

- Lecture(s):



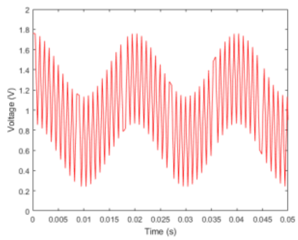
Cheating: Despiration/Ignorance is not an excuse…



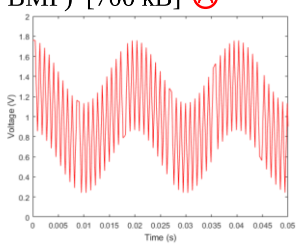
Platypus: File-Types & ~~DDoS~~

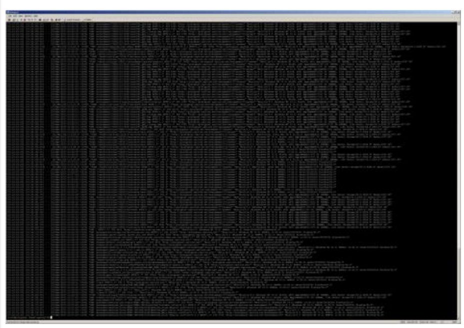
Please use appropriate filetypes

- **PNG [20 kB]** ✔



- (≠ BMP) [700 kB] ☹





Feedback on the Peer Review/Flagged Answers

Please Note

(1) "-1"

- Is an indicator in Platypus$_1$ that **nothing was calculated**.
- It does not effect grades at all (it's treated as a NAN)

(2) Flag "serious and egregious" oversights in the marking

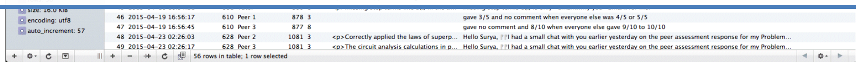
- "why so low", "give me mark plz"
  is not an egregious oversight

(3) If a peer or tutor gave you a lower than expected mark, then it might mean that you didn't communicate it clearly to them.

- Ask your self how you can do better?
- Remember: "Seeing is forgetting the name …"

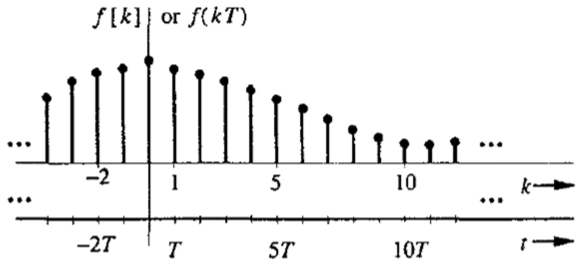
(4) Keep in mind the big picture here

- Focus on the learning, not the marks



Discrete-Time
Signal Analysis

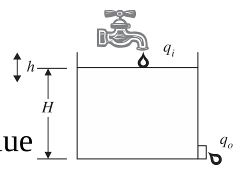
Discrete-Time Signal: *f[k]*



- Discrete-time signal:
  - May be denoted by $f(kT)$, where time t values are specified at $t = kT$
  - ***OR*** $f[k]$ and viewed as a function of k ($k \in$ integer)
- Continuous-time exponential:
  - $f(t) = e^{-t}$, sampled at $T = 0.1$ ➔ $f(kT) = e^{-kT} = e^{-0.1k}$

Why e^{-kT} ?

- Solution to First-Order ODE!
- Ex: “Tank” Fill
- Where:
  - H=steady-state fluid height in the tank
  - h=height perturbation from the nominal value
  - Q=steady-state flow rate through the tank
  - q_i=inflow perturbation from the nominal value
  - q_0=outflow perturbation from the nominal value

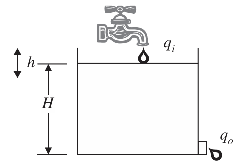


- Goal: Maintain H by adjusting Q.

Why e^{-kT} ? [2]

- $h = Rq_0$
- $\frac{dC(h+H)}{dt} = (q_i + Q) - (q_0 + Q)$
- $\frac{dh}{dt} + \frac{h}{\tau} = \frac{q_i}{C}$
- $\tau = RC$
- Solution:

$$h(t) = e^{\frac{t-t_0}{\tau}} h(t_0) + \frac{1}{C}\int_{t_0}^{t} e^{\frac{t-\lambda}{\tau}} q_i(\lambda) d\lambda$$



- For a fixed period of time (T) and steps k=0,1,2,…:

$$h(k+1) = e^{\frac{-T}{\tau}} h(k) + R\left[1 - e^{-\frac{T}{\tau}}\right] q_i(k)$$

So Why Is this a Concern? **Difference equations**

Difference equations arise in problems where the independent variable, usually time, is assumed to have a discrete set of possible values. The nonlinear difference equation

$$\begin{aligned} y(k+n) = f[&y(k+n-1),\ y(k+n-2),\ \ldots,\ y(k+1),\ y(k),\ u(k+n), \\ &u(k+n-1),\ \ldots,\ u(k+1),\ u(k)] \end{aligned} \tag{2.1}$$

with forcing function $u(k)$ is said to be of order n because the difference between the highest and lowest time arguments of $y(.)$ and $u(.)$ is n. The equations we deal with in this text are almost exclusively linear and are of the form

$$\begin{aligned} &y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) \\ &\quad = b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k) \end{aligned} \tag{2.2}$$

We further assume that the coefficients a_i, b_i, $i = 0, 1, 2, \ldots,$ are constant. The difference equation is then referred to as linear time invariant, or LTI. If the forcing function $u(k)$ is equal to zero, the equation is said to be *homogeneous*.

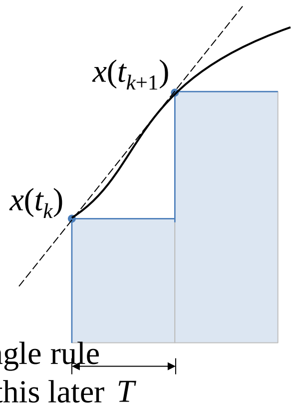
Difference equations can be solved using classical methods analogous to those available for differential equations. Alternatively, z-transforms provide a convenient approach for solving LTI equations, as discussed in the next section.

Euler's method*

- Dynamic systems can be approximated† by recognising that:

$$\dot{x} \cong \frac{x(k+1) - x(k)}{T}$$

- As $T \to 0$, approximation error approaches 0



*Also known as the forward rectangle rule
†Just an approximation – more on this later

Difference Equation: Euler's approximation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} \quad \Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} \approx \frac{x_{k+1} - x_k}{T}$$

For small enough T, this can be used to approximate a continuous controller by a discrete controller:

1. Laplace transform $\longrightarrow$ differential equation

   e.g.

$$D(s) = \frac{U(s)}{E(s)} = \frac{K(s+a)}{(s+b)} \quad \Longrightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}t} + b\,u = K\Big(\frac{de}{dt} + a\,e\Big)$$

2. Differential equation $\longrightarrow$ difference equation

   e.g.

$$\frac{u_{k+1} - u_k}{T} + b\,u_k = K\Big(\frac{e_{k+1} - e_k}{T} + a\,e_k\Big)$$

$$\Longrightarrow u_{k+1} = (1 - bT)u_k + K e_{k+1} + K(aT - 1)e_k$$
$$= -a_1 u_k + b_0 e_{k+1} + b_1 e_k$$

Difference Equation: Euler's approximation [2]

Discrete controller recurrence equation:

$$\boxed{u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \ldots + b_0 e_k + b_1 e_{k-1} + \ldots}$$

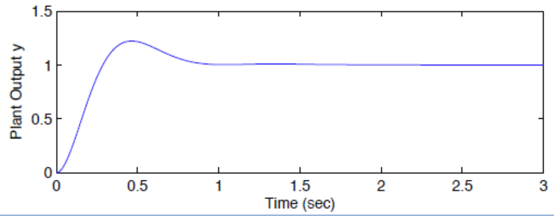
coefficients $a_1, a_2, \ldots,\ b_0, b_1, \ldots$ depend on T

Example

Controller: $D(s) = \dfrac{K(s+a)}{(s+b)}$, $K = 70,\ a = 2\ \text{rad s}^{-1},\ b = 10\ \text{rad s}^{-1}$

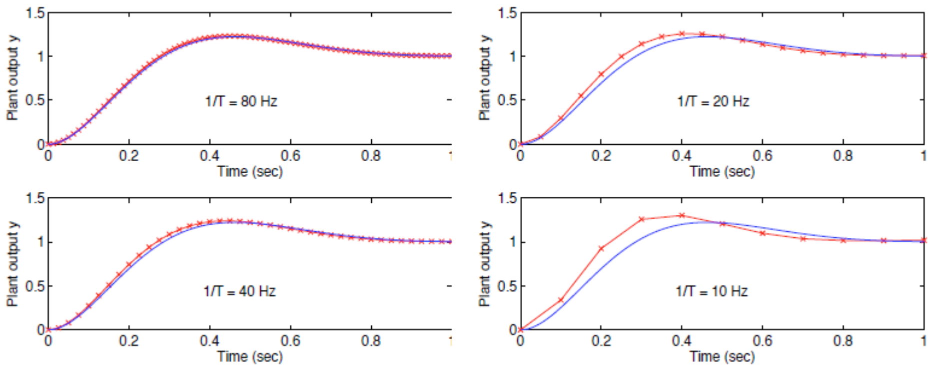
Plant: $G(s) = \dfrac{1}{s(s+1)}$

- Step response with continuous controller:



Difference Equation: Euler's approximation [3]

- Step responses with discrete controller:



Difference Equation: Euler's approximation [4]

- At high enough sample rates Euler's approximation works well:
  - discrete controller ≈ continuous controller

- **But** if sampling is not fast enough the approximation is poor:

$$\frac{1}{T} > 30 \times [System\ Bandwidth]$$

- Works, but Not Efficient (**η**)

- Later (May) We consider:
  - better ways of representing continuous systems in discrete-time
  - ways of analysing discrete controllers directly

Linear Differential System Order

$$Q(D)y(t) = P(D)f(t)$$

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$
$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0$$

→ *y(t)=P(D)/Q(D) f(t)*
P(D): M
Q(D): N
(yes, N is de**N**ominator)

- In practice: m ≤ n

∵ **if** m > *n:*
then the system is an
(m - n)th -order differentiator of high-frequency signals!

- Derivatives magnify noise!

Linear Differential Systems

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) =$$
$$b_m\frac{d^m f}{dt^m} + b_{m-1}\frac{d^{m-1}f}{dt^{m-1}} + \cdots + b_1\frac{df}{dt} + b_0 f(t) \qquad (2.1a)$$

where all the coefficients a_i and b_i are constants. Using operational notation D to represent d/dt, we can express this equation as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)\,y(t)$$
$$= (b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0)\,f(t) \qquad (2.1b)$$

or

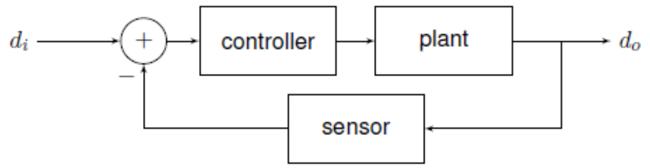
$$Q(D)y(t) = P(D)f(t) \qquad (2.1c)$$

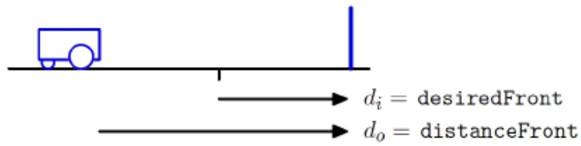
where the polynomials $Q(D)$ and $P(D)$ are

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 \qquad (2.2a)$$
$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0 \qquad (2.2b)$$

Discrete-Time
System Analysis

Simple Controller Goes Digital





plant: $y[n] = y[n-1] - Tu[n-1]$

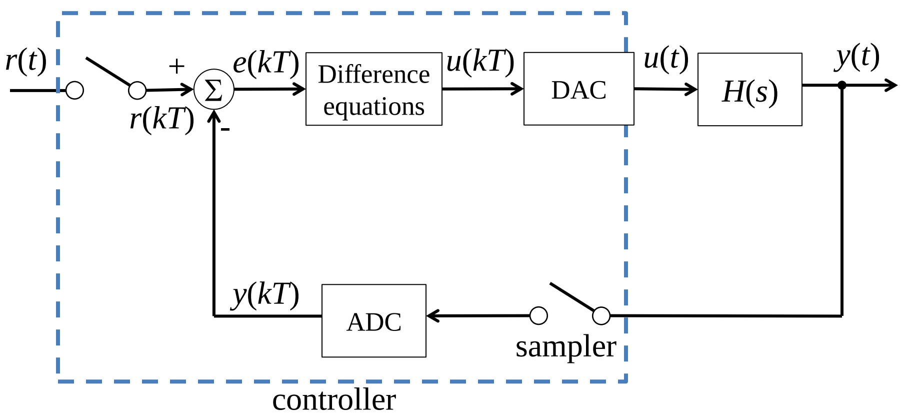
sensor: $y[n] = u[n-1]$

controller: $y[n] = Ku[n]$

Complex system behaviors, depending on K

Digitisation

- Continuous signals sampled with period T
- kth control value computed at $t_k = kT$



Digitisation

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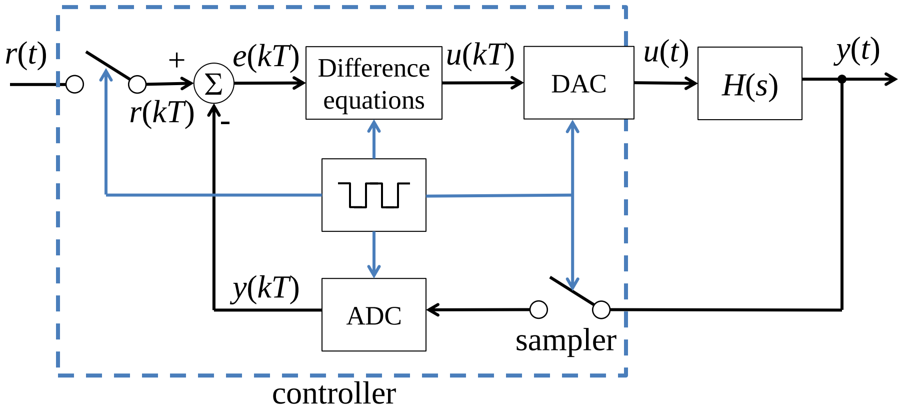
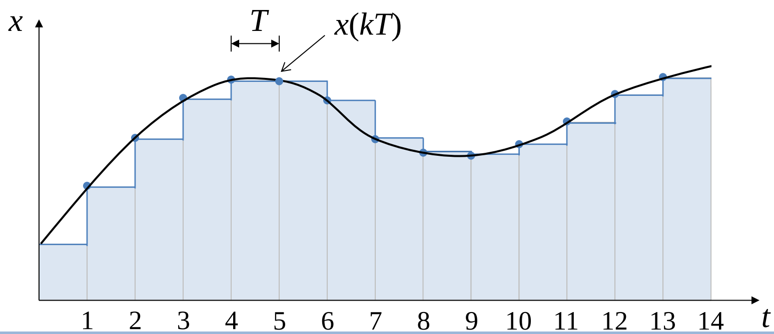


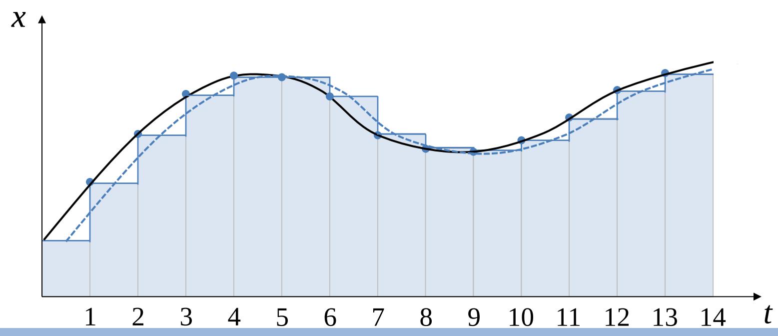
Return to the discrete domain

- Recall that continuous signals can be represented by a series of samples with period T



Zero Order Hold

- An output value of a synthesised signal is held constant until the next value is ready
  - This introduces an effective delay of $T/2$



Effect of ZOH Sampling

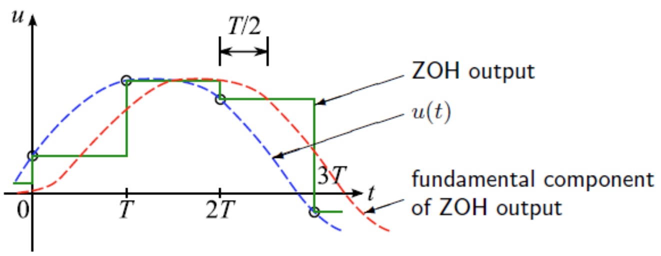
Lower sample rate $\implies$ more oscillatory response

— Why?

Sampling and reconstruction introduces:

delay in time domain
& phase lag in freq. domain $\leftarrow$ can destabilize the closed loop system

On average $u(kT)$ is delayed by $T/2$ relative to $u(t)$ due to the ZOH:



Effect of ZOH Sampling

The ZOH delay of $T/2$ (sec) causes

phase lag $= \omega T/2$ (rad) at ω rad s^{-1}

phase lag $= \pi/2 = 90^\circ$ at $\omega = \pi/T$ [= Nyquist rate]

phase lag $= \pi/30 = 6^\circ$ at $\omega = \pi/(15T)$

- ⋆ 90° phase lag could be catastrophic
- ⋆ If $\omega_{\text{samp}} > 30 \times \omega_{\max}$,

  then system bandwidth: $\omega_{\max} < \pi/(15T)$,

  so the maximum phase lag is less than 6°

  usually safe to ignore
- ⋆ Any time needed to compute u_k causes additional delay (!)

Back to the future

A quick note on causality:

- Calculating the "(k+1)th" value of a signal using

$$y(k+1) = \underbrace{x(k+1)}_{\text{future value}} + \underbrace{Ax(k) - By(k)}_{\text{current values}}$$

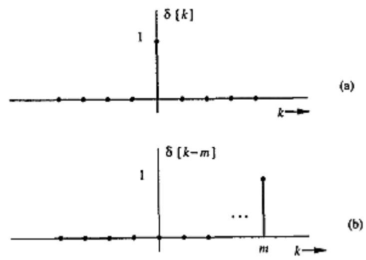
relies on also knowing the next (future) value of $x(t)$.

(this requires very advanced technology!)

- Real systems always run with a delay:

$$y(k) = x(k) + Ax(k-1) - By(k-1)$$

Discrete-Time Impulse Function $\delta[k]$



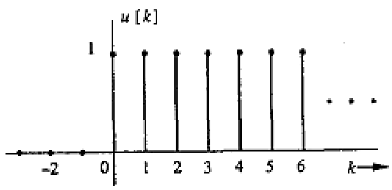
The discrete-time counterpart of the continuous-time impulse function $\delta(t)$ is $\delta[k]$, defined by

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{8.1}$$

This function, also called the unit impulse sequence, is shown in Fig. 8.3a. The time-shifted impulse sequence $\delta[k-m]$ is depicted in Fig. 8.3b. Unlike its continuous-time counterpart $\delta(t)$, this is a very simple function without any mystery.

Later, we shall express an arbitrary input $f[k]$ in terms of impulse components. The (zero-state) system response to input $f[k]$ can then be obtained as the sum of system responses to impulse components of $f[k]$.

Discrete-Time Unit Step Function $u[k]$



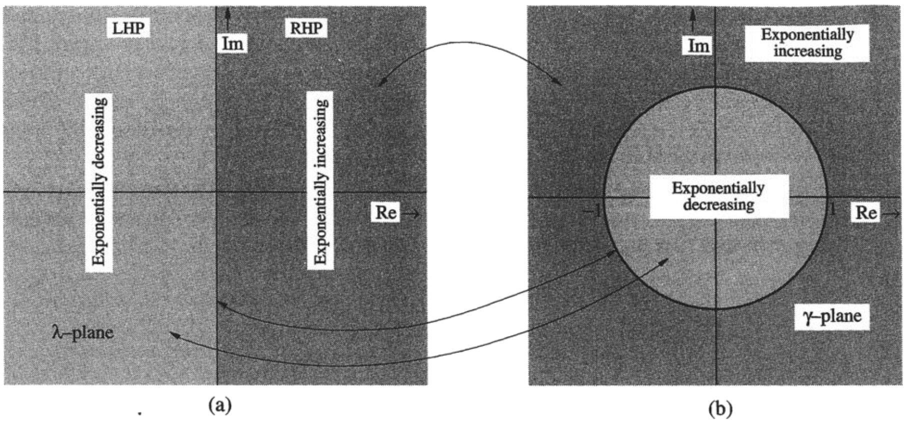
The discrete-time counterpart of the unit step function $u(t)$ is $u[k]$ (Fig. 8.4), defined by

$$u[k] = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases} \tag{8.2}$$

If we want a signal to start at $k = 0$ (so that it has a zero value for all $k < 0$), we need only multiply the signal with $u[k]$.

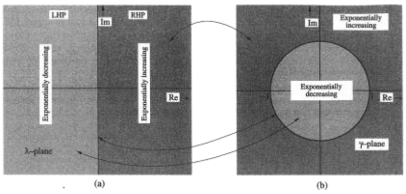
Discrete-Time Exponential γ^k

$$e^{\lambda k} = \gamma^k$$



Discrete-Time Exponential γ^k

- $e^{\lambda k} = \gamma^k$
- $\gamma = e^{\lambda}$ or $\lambda = \ln \gamma$

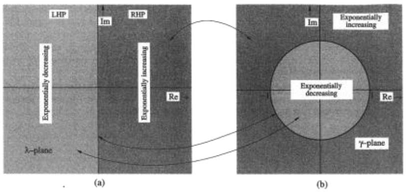


- In discrete-time systems, unlike the continuous-time case, the form γ^k proves more convenient than the form $e^{\lambda k}$

Why?

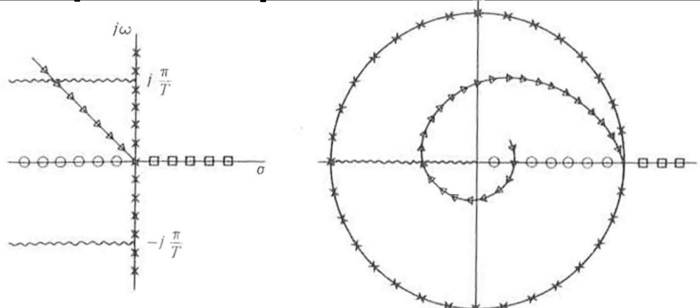
- Consider $e^{j\Omega k}$ ($\lambda = j\Omega$ ∴ constant amplitude oscillatory)
- $e^{j\Omega k}$ ➔ γ^k, for $\gamma \equiv e^{j\Omega}$
- $|e^{j\Omega}| = 1$, hence $|\gamma| = 1$

Discrete-Time Exponential γ^k



- Consider $e^{\lambda k}$
  When λ: LHP
- Then
- $\gamma = e^{\lambda}$
- $\gamma = e^{\lambda} = e^{a+jb} = e^{a}e^{jb}$
- $|\gamma| = \left|e^{a}e^{jb}\right| = |e^{a}| \because \left|e^{jb}\right| = 1$

Hint: Use γ to Transform s ↔ z: $z=e^{sT}$



s-plane	Symbol	z-plane
$s = j\omega$ Real frequency axis	× × ×	$\lvert z\rvert = 1$ Unit circle
$s = \sigma \geq 0$	□□□	$z = r \geq 1$
$s = \sigma \leq 0$	○○○	$z = r,\ 0 \leq r \leq 1$
$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -a + jb$ Constant damping ratio if ζ is fixed and ω_n varies	△△△	$z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T) = e^{-aT}$, $\theta = \omega_n T\sqrt{1-\zeta^2} = bT$ Logarithmic spiral
$s = \pm j(\pi/T) + \sigma, \sigma \leq 0$	〰〰	$z = -r$

BREAK

z Transforms

(Digital Systems Made eZ)

Review and Extended Explanation

The z-transform

- The discrete equivalent is the z-Transform†:

$$\mathcal{Z}\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$$

and

$$\mathcal{Z}\{f(k-1)\} = z^{-1} F(z)$$

$$x(k) \longrightarrow \boxed{F(z)} \longrightarrow y(k)$$

Convenient!

†This is not an approximation, but approximations are easier to derive

The z-Transform

- It is defined by:

$$z = re^{j\omega}$$

Or in the Laplace domain:

$$z = e^{sT}$$

- Thus: $Y(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$ or $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$

- I.E., It's a discrete version of the Laplace:

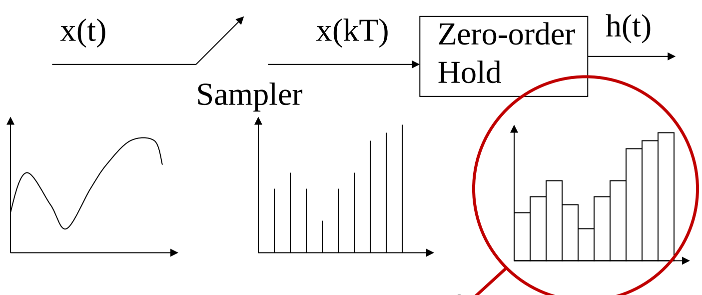
$$f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{z}{z - e^{-aT}}$$

The z-transform

- In practice, you'll use look-up tables or computer tools (ie. Matlab) to find the *z*-transform of your functions

$F(s)$	$F(kt)$	$F(z)$
$\frac{1}{s}$	1	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\frac{1}{s^2+a^2}$	$\sin(akT)$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$

Zero-order-hold (ZOH)



- Assume that the signal x(t) is zero for t<0, then the output h(t) is related to x(t) as follows:

$$h(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] + \cdots$$
$$= \sum_{k=0}^{\infty} x(kT)[1(t-kT) - 1(t-(k+1)T)]$$

Transfer function of Zero-order-hold (ZOH)

- Recall the Laplace Transforms ($\mathcal{L}$) of:

$$\mathcal{L}[\delta(t)] = 1 \qquad \mathcal{L}[f(t-kT)] = F(s)e^{-kTs}$$

$$\mathcal{L}\left[\delta(t-kT)\right] = e^{-kTs} \qquad \mathcal{L}[1(t-kT)] = \frac{e^{-kTs}}{s}$$

- Thus the $\mathcal{L}$ of h(t) becomes:

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Transfer function of Zero-order-hold (ZOH)

… Continuing the $\mathcal{L}$ of h(t) …

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\rightarrow X(s) = \mathcal{L}\left[\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right] = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\therefore H(s) = \mathcal{L}[h(t)] = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}X(s)$$

➔ Thus, giving the transfer function as:

$$\boxed{G_{\text{ZOH}}(s) = \frac{H(s)}{X(s)} = \frac{1-e^{-Ts}}{s}} \xrightarrow{\mathcal{Z}} \boxed{G_{ZOH}(z) = \frac{\left(1-e^{-aT}\right)}{z-e^{-aT}}}$$

Coping with Complexity

Transfer functions help control complexity

– Recall the Laplace transform:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

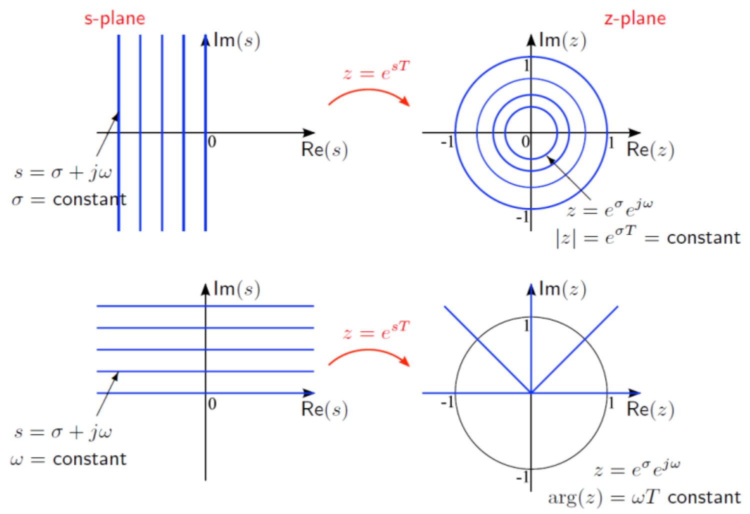
where

$$\mathcal{L}\{\dot{f}(t)\} = sF(s)$$

$$x(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t)$$

- Is there a something similar for sampled systems?

S-Plane to z-Plane [1/2]



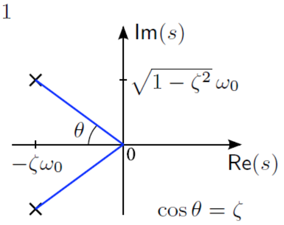
S-Plane to z-Plane [2/2]

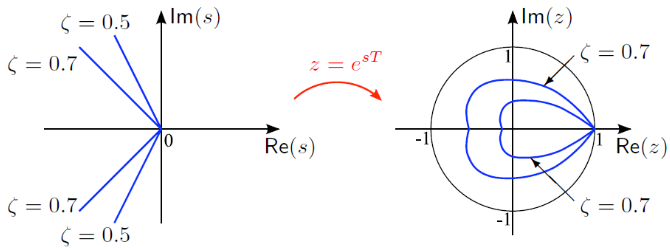
Pole locations for constant damping ratio $\zeta < 1$

$$s^2 + \zeta\omega_0 s + \omega_0^2 = 0$$

$$\Downarrow$$

$$s = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\,\omega_0$$





$s = -\zeta\omega_0 + j\sqrt{1-\zeta^2}\,\omega_0$: ζ = constant $\qquad z = e^{-\zeta\omega_0 T} e^{-j\sqrt{1-\zeta^2}\omega_0 T}$

Relationship with s-plane poles and z-plane transforms

If $F(s)$ has a pole at $s = a$
then $F(z)$ has a pole at $z = e^{aT}$

$\uparrow$

consistent with $z = e^{sT}$

What about transfer functions?

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

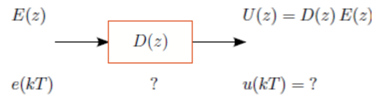
$\downarrow$

If $G(s)$ has poles $s = a_i$
then $G(z)$ has poles $z = e^{a_i T}$

but the zeros are unrelated

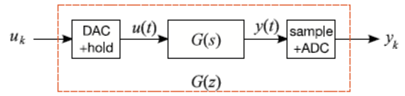
$\mathcal{F}(s)$	$f(kT)$	$F(z)$
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-1}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$
$\frac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$

$s \leftrightarrow z$: Pulse Transfer Function Models



- Pulse in Discrete is equivalent to Dirac-δ

$$e_k = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$



➔

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}_{t=kT}\right\} = \boldsymbol{(1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}}$$

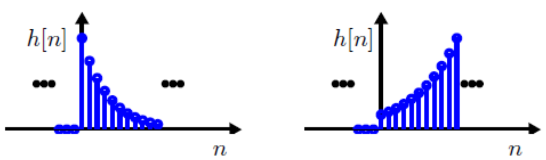
Source: Oxford 2A2 Discrete Systems, Tutorial Notes p. 26

z-Transforms for Difference Equations

- First-order linear constant coefficient difference equation:

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$



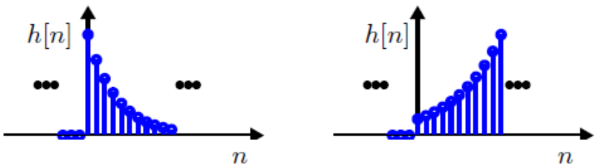
$$h[n] = \begin{cases} ba^n & n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H(z) = \sum_{k=0}^{\infty} ba^k z^{-k} = b\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k = \frac{b}{1 - az^{-1}}, \quad \text{when } |z| > |a|.$$

z-Transforms for Difference Equations

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$



$$y[n] - ay[n-1] = bu[n]$$
$$\updownarrow$$
$$Y(z) - az^{-1}Y(z) = bU(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1 - az^{-1}},$$ when does it converge?

Properties of the the z-transform

- Some useful properties
  - **Delay by n samples**: $\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$
  - **Linear**: $\mathcal{Z}\{af(k) + bg(k)\} = \mathrm{a}F(z) + bG(z)$
  - **Convolution**: $\mathcal{Z}\{f(k) * g(k)\} = F(z)G(z)$

So, all those block diagram manipulation tools you know and love will work just the same!

The z-Transform

- It is defined by:

$$z = re^{j\omega}$$

- Or in the Laplace domain:

$$z = e^{sT}$$

- That is → it is a discrete version of the Laplace:

$$f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{z}{z - e^{-aT}}$$

The z-Transform [2]

- Thus:

$$Y(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k} \qquad y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$$

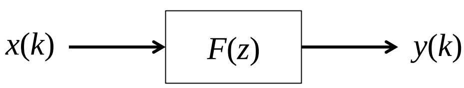
- z-Transform is analogous to other transforms:

$$\mathcal{Z}\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$$

and

$$\mathcal{Z}\{f(k-1)\} = z^{-1}F(z)$$

∴ Giving:



The z-Transform [3]

- The z-Transform may also be considered from the Laplace transform of the impulse train representation of sampled signal

$$u^*(t) = u_0\delta(t) + u_1\delta(t-T) + \ \dots + u_{k(t-kT)} + \ \dots$$

$$= \sum_{k=0}^{\infty} u_k \delta(t-kT)$$

$$U^*(s) = u_0 + u_1 e^{-sT} + \cdots + u_k e^{-skT} + \cdots$$

$$= \sum_{k=0}^{\infty} u_k\, e^{-ksT}$$

$$U(z) = \sum_{k=0}^{\infty} u_k\, z^{-k}, \qquad z = e^{sT}$$

The z-transform

- In practice, you'll use look-up tables or computer tools (ie. Matlab) to find the *z*-transform of your functions

$F(s)$	$F(kt)$	$F(z)$
$\frac{1}{s}$	1	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\frac{1}{s^2+a^2}$	$\sin(akT)$	$\frac{z \sin aT}{z^2 - (2\cos aT)z + 1}$

z-Transform Example

- Obtain the z-Transform of the sequence:

$$x[k] = \{3, 0, 1, 4,1,5, \dots\}$$

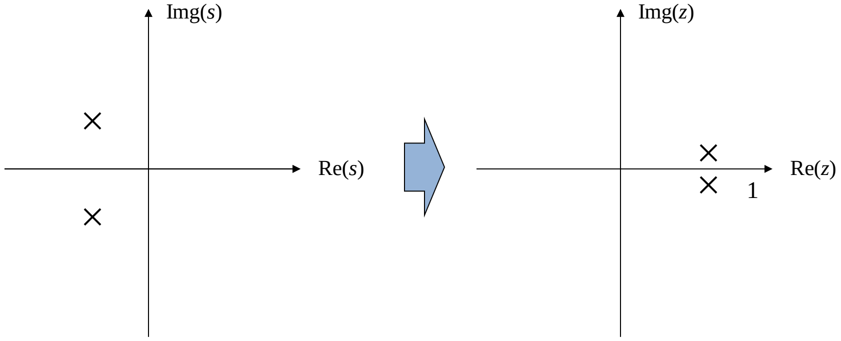
- Solution:

$$X(z) = 3 + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5}$$

The z-Plane

z-domain poles and zeros can be plotted just like s-domain poles and zeros (of the $\mathcal{L}$):

- S-plane:
- $\boldsymbol{z = e^{sT}}$ Plane

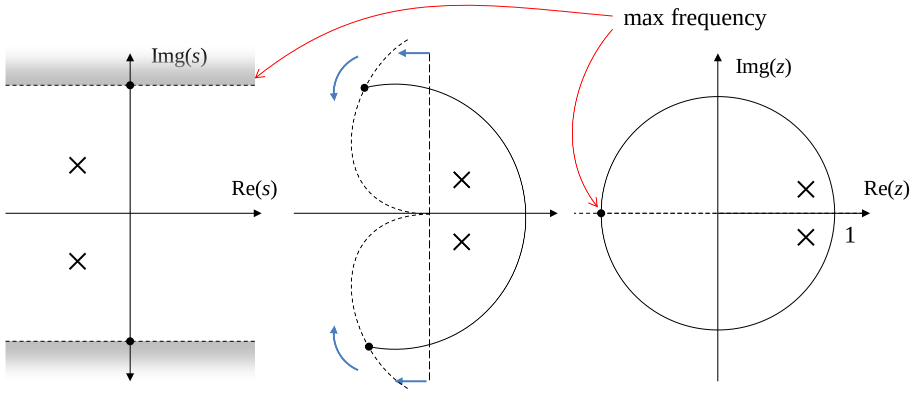


– λ – Plane

– γ – Plane

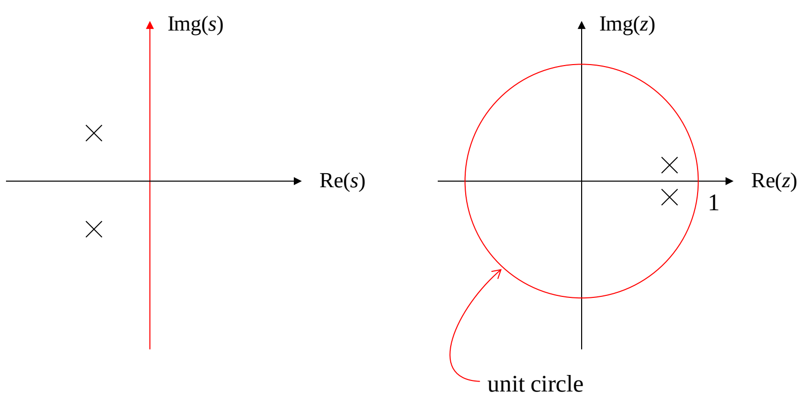
Deep insight #1

The mapping between continuous and discrete poles and zeros acts like a distortion of the plane



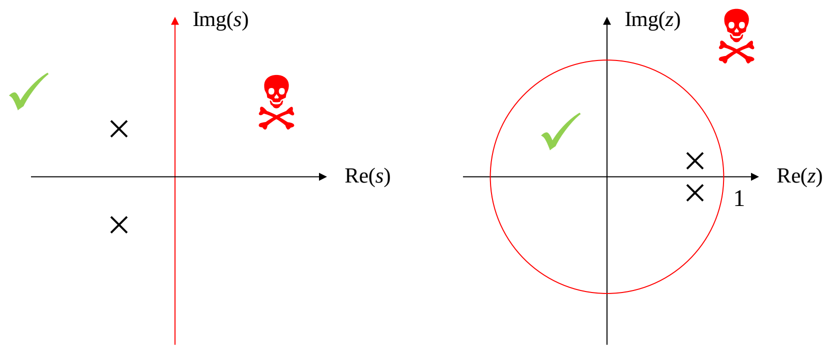
γ-plane Stability

- For a γ-Plane (e.g. the one the z-domain is embedded in) the unit circle is the system stability bound



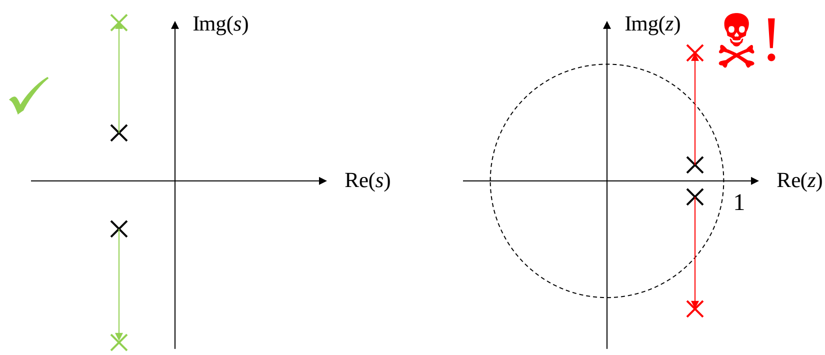
γ-plane Stability

- That is, in the *z*-domain,
  the unit circle is the system stability bound



z-plane stability

- The z-plane root-locus in closed loop feedback behaves just like the s-plane:



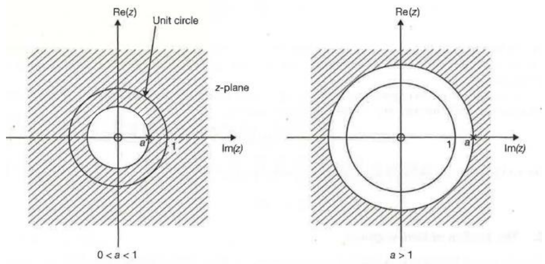
Region of Convergence

- For the convergence of X(z) we require that

$$\sum_{n=0}^{\infty} \left| az^{-1} \right|^n < \infty$$

- Thus, the ROC is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Then

$$X(z) = \frac{z}{z-a} \qquad |z| > |a|$$



An example!

- Back to our difference equation:

$$y(k) = x(k) + Ax(k-1) - By(k-1)$$

becomes

$$Y(z) = X(z) + Az^{-1}X(z) - Bz^{-1}Y(z)$$
$$(z+B)Y(z) = (z+A)X(z)$$

which yields the transfer function:

$$\frac{Y(z)}{X(z)} = \frac{z+A}{z+B}$$

Note: It is also not uncommon to see systems expressed as polynomials in z^{-n}

This looks familiar…

- Compare:

$$\frac{Y(s)}{X(s)} = \frac{s+2}{s+1} \text{ vs } \frac{Y(z)}{X(z)} = \frac{z+A}{z+B}$$

How are the Laplace and *z* domain representations related?

Linearity:

$$a_1 y_1[n] + a_2 y_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} a_1 Y_1(z) + a_2 Y_2(z)$$

Z-Transform Properties: Time Shifting

$$y[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} Y(z)$$

$$\begin{aligned} y_2[n] &= y[n-n_0] \\ Y_2(e^{jw}) &= \sum_{k=-\infty}^{\infty} y[k-n_0] z^{-k} \\ &= \sum_{l=-\infty}^{\infty} y[l] z^{-(l+n_0)} \\ &= z^{-n_0} Y(z) \end{aligned}$$

- Two Special Cases:
- z^{-1}: the *unit-delay operator:*

$$x[n-1] \leftrightarrow z^{-1} X(z) \qquad R' = R \cap \{0 < |z|\}$$

- z: *unit-advance operator:*

$$x[n+1] \leftrightarrow zX(z) \qquad R' = R \cap \{|z| < \infty\}$$

More Z-Transform Properties

- Time Reversal

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \qquad R' = \frac{1}{R}$$

- Multiplication by z^n

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad R' = |z_0| R$$

- Multiplication by n (or Differentiation in z):

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \qquad R' = R$$

- Convolution

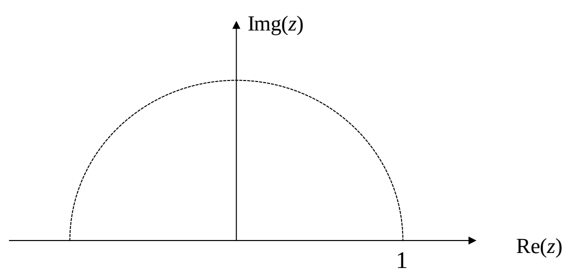
$$x_1[n] \leftrightarrow X_1(z) \qquad \text{ROC} = R_1$$
$$x_2[n] \leftrightarrow X_2(z) \qquad \text{ROC} = R_2$$

$$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \qquad R' \supset R_1 \cap R_2$$

The z-plane [for all pole systems]

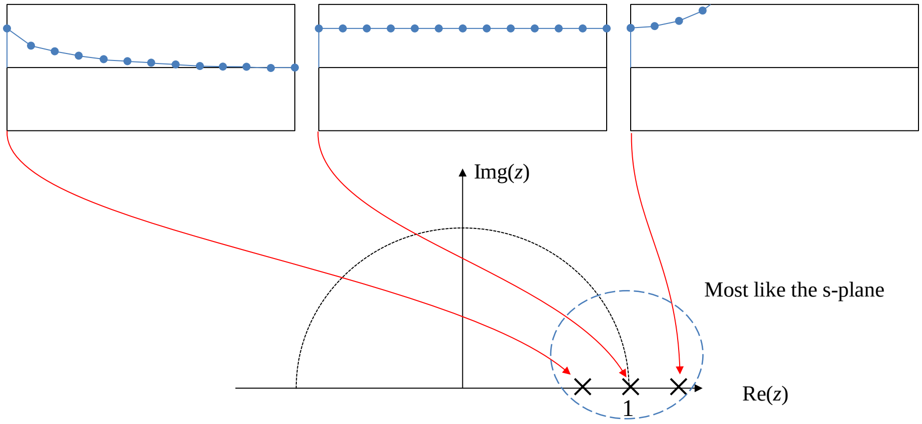
- We can understand system response by pole location in the *z*-plane

[Adapted from Franklin, Powell and Emami-Naeini]



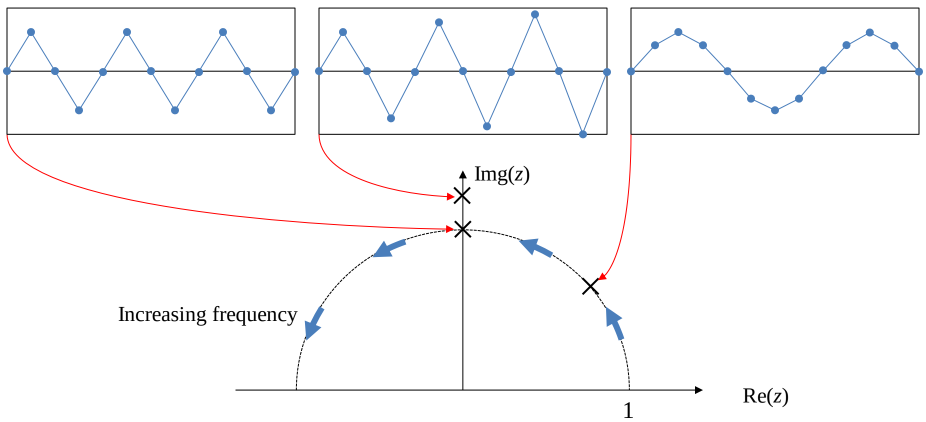
Effect of pole positions

- We can understand system response by pole location in the *z*-plane



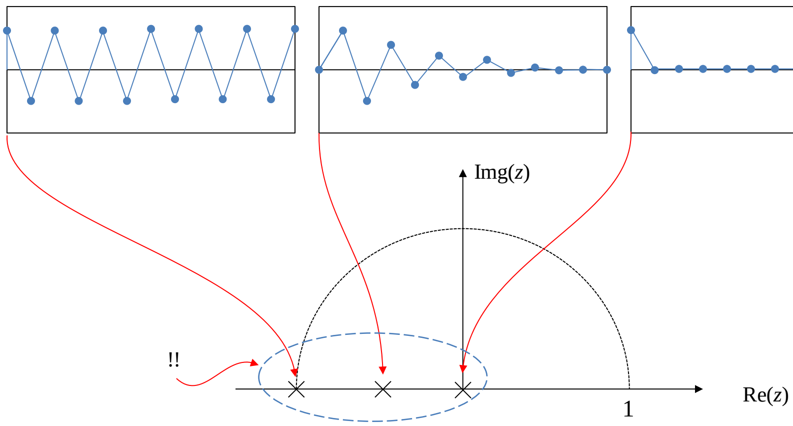
Effect of pole positions

- We can understand system response by pole location in the *z*-plane



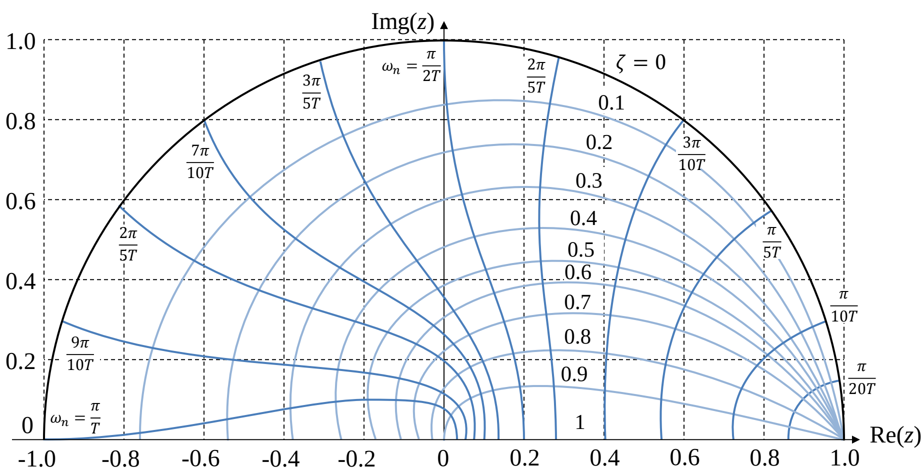
Effect of pole positions

- We can understand system response by pole location in the *z*-plane



z-Plane Response for 2nd Order Systems: Damping (ζ) and Natural frequency (ω)

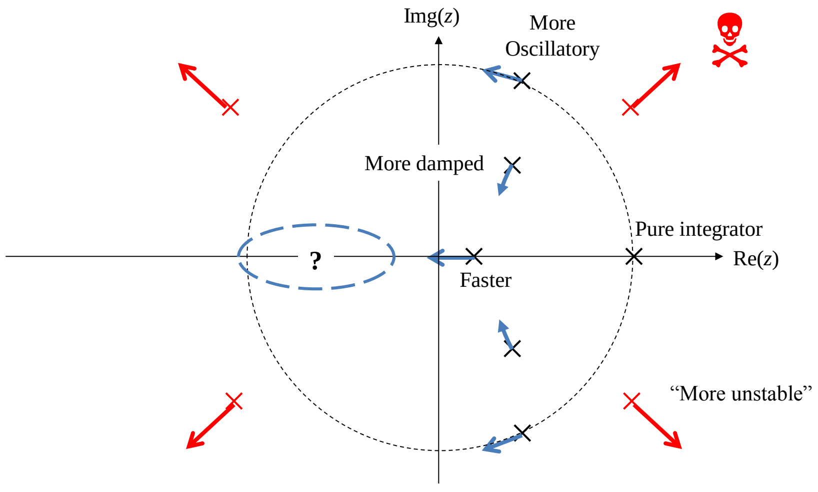
$$z = e^{sT} \text{ where } s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



[Adapted from Franklin, Powell and Emami-Naeini]

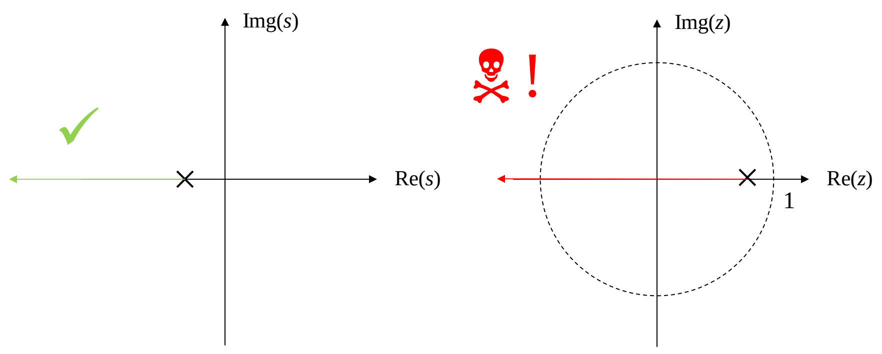
Recall dynamic responses

- Ditto the z-plane:



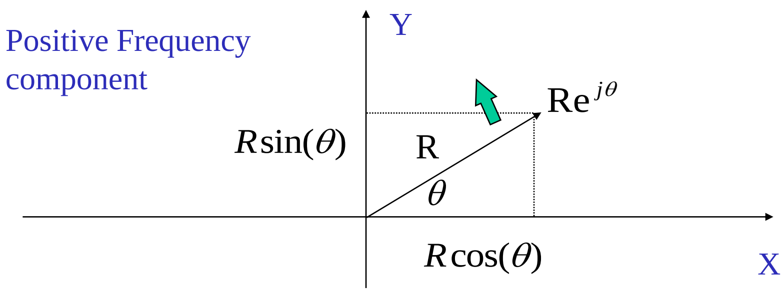
Deep insight #2

- Gains that stabilise continuous systems can actually ***destabilise*** digital systems!



Sampling & ANTIALIASING (Recap)

SaV (Signals as Vectors): Signals as Complex Numbers → Phasors

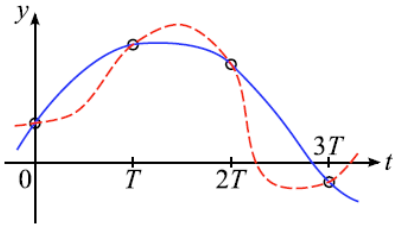


$$\begin{aligned} Re^{j\theta} &= (R\cos\theta, R\sin\theta) \\ &= R\cos\theta + jR\sin\theta \\ &= R(\cos\theta + j\sin\theta) \end{aligned}$$

Nyquist sampling theorem

What continuous signal is represented by a given set of samples?

Infinitely many continuous signals have the same discrete samples:



An answer is provided by Nyquist's sampling theorem:

> A signal $y(t)$ is uniquely defined by its samples $y(kT)$ if the sampling frequency is more than twice the bandwidth of $y(t)$.

Nyquist sampling theorem [2]

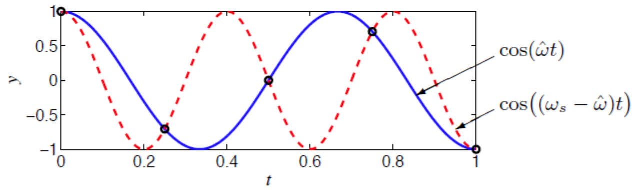
Example – Sampled sinusoidal signal

Sample $\cos(\hat{\omega}t)$ at frequency $\omega_s = 2\pi/T$:

$$y(t) = \cos(\hat{\omega}t) \xrightarrow{\text{sample}} y(kT) = \cos(k\,\hat{\omega}T) = \cos(2\pi k\,\hat{\omega}/\omega_s)$$

Identical samples are obtained from a sinusoid with frequency $\omega_s - \hat{\omega}$:

$$\cos\big((\omega_s - \hat{\omega})t\big) \xrightarrow{\text{sample}} \cos\big(k(\omega_s - \hat{\omega})T\big) = \cos(2\pi k - 2\pi k\,\hat{\omega}/\omega_s)$$
$$= \cos(2\pi k\,\hat{\omega}/\omega_s)$$



The spectrum of $y(kT)$ contains an alias at frequency $\omega_s - \hat{\omega}$!!

↑

(a copy of the original signal $y(t)$ shifted to a different frequency)

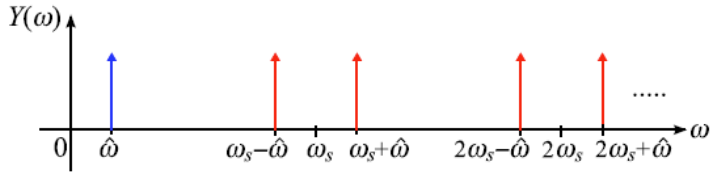
Nyquist sampling theorem & alliasing

Example – Sampled sinusoidal signal

By the same argument, $y(kT)$ contains an infinite number of aliases at

$$\omega_s \pm \hat{\omega},\ 2\omega_s \pm \hat{\omega},\ 3\omega_s \pm \hat{\omega}, \ldots$$

spectrum of $y(kT)$:



The Nyquist sampling theorem requires $\omega_s > 2\hat{\omega}$

$$\Downarrow$$

$y(t)$ and alias spectra do not overlap
$y(t)$ can be recovered without distortion from $y(kT)$ (via low-pass filter)

Aliasing: Nonuniqueness of Discrete-Time Sinusoids [p. 553]

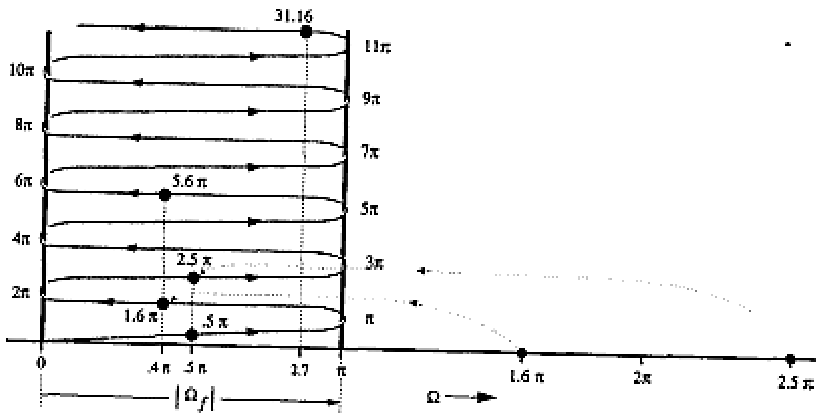
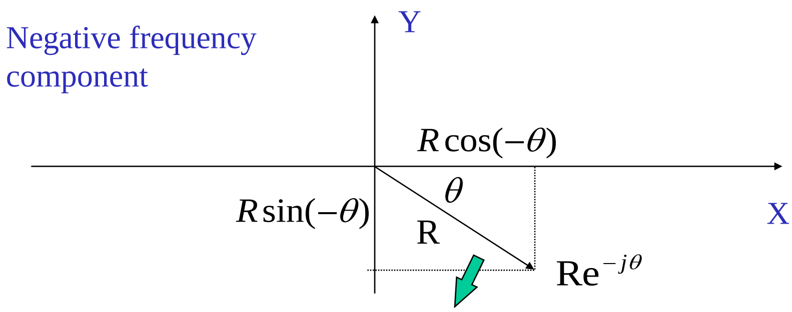


Fig. 8.11 A graphical artifice to determine the reduced frequency of a discrete-time sinusoid.

Complex Numbers and Phasors



$$
\begin{aligned}
\mathrm{Re}^{-j\theta} &= (R\cos(-\theta), R\sin(-\theta)) \\
&= R\cos(-\theta) + jR\sin(-\theta) \\
&= R(\cos\theta - j\sin\theta)
\end{aligned}
$$

Positive and Negative Frequencies

- Frequency is the derivative of phase
  more nuanced than :

$$\frac{1}{\tau} = repetition\ rate$$

- Hence both positive and negative frequencies are possible.

- Compare
  - velocity vs speed
  - frequency vs repetition rate

Negative Frequency

- Q: What is negative frequency?
- A: A mathematical convenience

- Trigonometrical FS
  - periodic signal is made up from
  - sum 0 to ∞ of sine and cosines ‘harmonics’

- Complex Fourier Series & the Fourier Transform
  - use $\exp(\ j\omega t)$ instead of $\cos(\omega t)$ and $\sin(\omega t)$
  - signal is sum from 0 to ∞ of $\exp(\pm j\omega t)$
  - same as sum -∞ to ∞ of $\exp(-j\omega t)$
  - which is more compact (i.e., less LaTeX!)

Next Time…

- **Digital Systems**

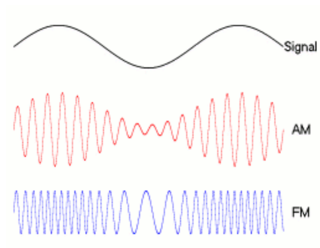
- Review:
  - Chapter 8 of Lathi

- A signal has many signals ☺
  [Unless it’s bandlimited. Then there is the one ω]

Modulation

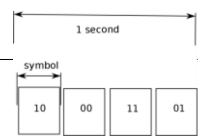
Analog Methods:

- AM - Amplitude modulation
  - Amplitude of a (carrier) is modulated to the (data)
- FM - Frequency modulation
  - Frequency of a (carrier) signal is varied in accordance to the amplitude of the (data) signal
- PM – Phase Modulation



Source: http://en.wikipedia.org/wiki/Modulation

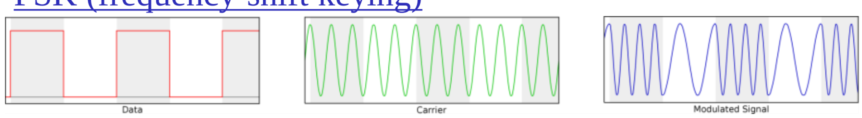
Modulation [Digital Methods]



Start with a "**symbol**" & place it on a channel

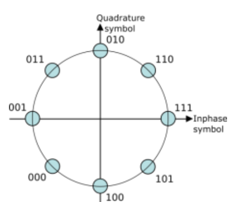
- ASK (amplitude-shift keying)

- FSK (frequency-shift keying)



- PSK (phase-shift keying)
- QAM (quadrature amplitude modulation)

$s(t) = A \cdot cos(\omega_c + \phi_i(t))$
$= x_i(t)\cos(\omega_c t) + x_q(t)\sin(\omega_c t)$



Source: http://en.wikipedia.org/wiki/Modulation | http://users.ecs.soton.ac.uk/sqc/EL334 | http://en.wikipedia.org/wiki/Constellation_diagram

Modulation [Example – V.32bis Modem]

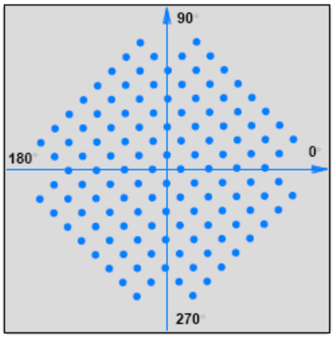
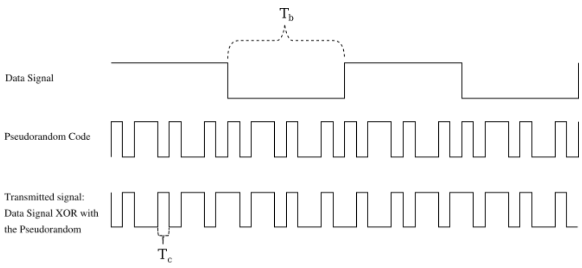


Figure 10.13 Illustration of the QAM constellation for a V.32bis dialup modem.

Source: Computer Networks and Internets, 5e, Douglas E. Comer

Multiple Access (Channel Access Method)

- Send multiple signals on 1 to N channel(s)
  - Frequency-division multiple access (FDMA)
  - Time-division multiple access (TDMA)
  - Code division multiple access (CDMA)
  - Space division multiple access (SDMA)
- CDMA:
  - Start with a pseudorandom code (the noise doesn't know your code)



Source: http://en.wikipedia.org/wiki/Code_division_multiple_access