http://elec3004.com

Digital Control Design

ELEC 3004: **Systems**: Signals & Controls
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Lecture 19

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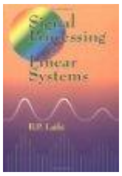
May 16, 2017

Lecture Schedule:

Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Systems: Linear Differential Systems
3	14-Mar	Sampling Theory & Data Acquisition
	16-Mar	Aliasing & Antialiasing
4	21-Mar	Discrete Time Analysis & Z-Transform
	23-Mar	Second Order LTID (& Convolution Review)
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
6	4-Apr	Digital Filters (IIR) & Filter Analysis
	6-Apr	Digital Filter (FIR)
7	11-Apr	Digital Windows
	13-Apr	FFT
	18-Apr	**Holiday**
	20-Apr	
	25-Apr	
8	27-Apr	Active Filters & Estimation
9	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	PID & State-Space
	11-May	State-Space Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response
	25-May	Applications in Industry
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

G. Franklin,
J. Powell,
M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990
[Available as UQ Ebook]

Today

➔ **State-space** ➔ **[A stately idea! ☺]**

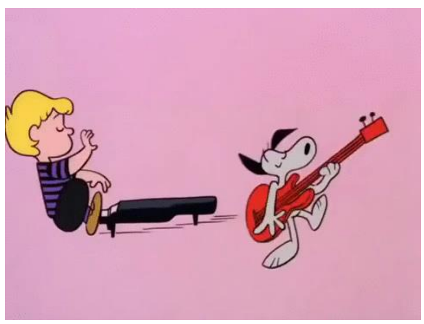
- FPW
  - Chapter 4: Discrete Equivalents to Continuous
  - Transfer Functions: The Digital Filter

- Lathi Ch. 13
  - § 13.2 Systematic Procedure for Determining State Equations
  - § 13.3 Solution of State Equations

Next Time

Final Exam Information announcement

- Date:
  Saturday, June/10
  (remember buses on Saturday Schedule)



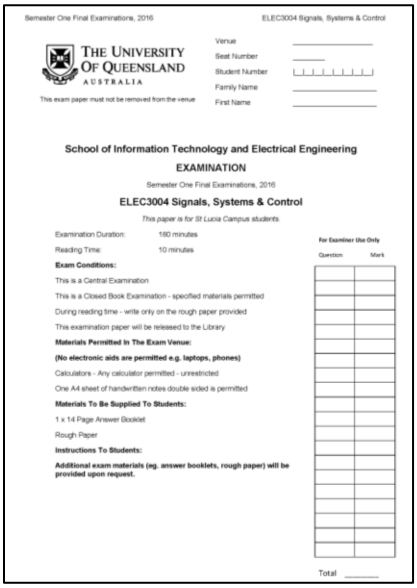
- Time:
  4:30-7:45 (+/-)

- Location:
  TBA

- UQ Exams are now "ID Verified"
  ➔ Please remember your ID! ➔

Final Exam Information

- Section 1:
  - Digital Linear Dynamical Systems
  - 5 Questions
  - 60 Points (33 %)
- Section 2:
  - Digital Processing / Filtering of Signals
  - 5 Questions
  - 60 Points (33 %)
- Section 3:
  - Digital & State-Space Control
  - 5 Questions
  - 60 Points (33 %)



Semester One Final Examinations, 2016 ELEC3004 Signals, Systems & Control

THE UNIVERSITY OF QUEENSLAND AUSTRALIA

This exam paper must not be removed from the venue

Venue ______

Seat Number ______

Student Number ______

Family Name ______

First Name ______

School of Information Technology and Electrical Engineering

EXAMINATION

Semester One Final Examinations, 2016

ELEC3004 Signals, Systems & Control

This paper is for St Lucia Campus students.

Examination Duration: 180 minutes

Reading Time: 10 minutes

Exam Conditions:

This is a Central Examination

This is a Closed Book Examination - specified materials permitted

During reading time - write only on the rough paper provided

This examination paper will be released to the Library

Materials Permitted In The Exam Venue:

(No electronic aids are permitted e.g. laptops, phones)

Calculators - Any calculator permitted - unrestricted

One A4 sheet of handwritten notes double sided is permitted

Materials To Be Supplied To Students:

1 x 14 Page Answer Booklet

Rough Paper

Instructions To Students:

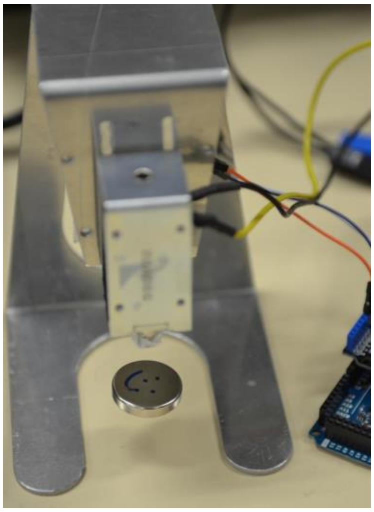
Additional exam materials (eg. answer booklets, rough paper) will be provided upon request.

For Examiner Use Only

Question	Mark

Total ______

NEXT WEEK: Lab 4 – LeviLab II:



- AKA "Revenge of the TUNING!"

Lab 4 News: Digital PID Controls

(AKA: Magic "PID Made Easy" Equations)

Implementation of Digital PID Controllers

We will consider the PID controller with an s-domain transfer function

$$\frac{U(s)}{X(s)} = G_c(s) = K_P + \frac{K_I}{s} + K_D s. \qquad (13.54)$$

We can determine a digital implementation of this controller by using a discrete approximation for the derivative and integration. For the time derivative, we use the **backward difference rule**

$$u(kT) = \left.\frac{dx}{dt}\right|_{t=kT} = \frac{1}{T}(x(kT) - x[(k-1)T]). \qquad (13.55)$$

The z-transform of Equation (13.55) is then

$$U(z) = \frac{1 - z^{-1}}{T}X(z) = \frac{z-1}{Tz}X(z).$$

The integration of $x(t)$ can be represented by the **forward-rectangular integration** at $t = kT$ as

$$u(kT) = u[(k-1)T] + Tx(kT), \qquad (13.56)$$

Source: Dorf & Bishop, Modern Control Systems, §13.9, pp. 1030-1

Implementation of Digital PID Controllers (2)

where $u(kT)$ is the output of the integrator at $t = kT$. The z-transform of Equation (13.56) is

$$U(z) = z^{-1}U(z) + TX(z),$$

and the transfer function is then

$$\frac{U(z)}{X(z)} = \frac{Tz}{z-1}.$$

Hence, the z-domain transfer function of the **PID controller** is

$$\boxed{G_c(z) = K_P + \frac{K_I T z}{z-1} + K_D \frac{z-1}{Tz}.} \quad (13.57)$$

The complete difference equation algorithm that provides the PID controller is obtained by adding the three terms to obtain [we use $x(kT) = x(k)$]

$$\begin{aligned} u(k) &= K_P x(k) + K_I[u(k-1) + Tx(k)] + (K_D/T)[x(k) - x(k-1)] \\ &= [K_P + K_I T + (K_D/T)]x(k) - K_D T x(k-1) + K_I u(k-1). \end{aligned} \quad (13.58)$$

Equation (13.58) can be implemented using a digital computer or microprocessor. Of course, we can obtain a PI or PD controller by setting an appropriate gain equal to zero.

Source: Dorf & Bishop, Modern Control Systems, §13.9, pp. 1030-1

Back to State-Space …

Solving State Space

Great, so how about control?

- Given $\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$, if we know $\mathbf{F}$ and $\mathbf{G}$, we can design a controller $u = -\mathbf{K}\boldsymbol{x}$ such that

$$\mathrm{eig}(\mathbf{F} - \mathbf{GK}) < 0$$

- In fact, if we have full measurement and control of the states of $\boldsymbol{x}$, we can position the poles of the system in arbitrary locations!

(Of course, that never happens in reality.)

Solving State Space…

- Recall:

$$\dot{x} = f\left(x, u, t\right)$$

- For Linear Systems:

$$\dot{x}\left(t\right) = A\left(t\right)x\left(t\right) + B\left(t\right)u\left(t\right)$$
$$y\left(t\right) = C\left(t\right)x\left(t\right) + D\left(t\right)u\left(t\right)$$

- For LTI:

$$\rightarrow \dot{x} = Ax + Bu$$
$$\rightarrow y = Cx + Du$$

➔ Solutions to State Equations

$$\dot{x} = Ax + Bu$$
$$sX(s) - x(0) = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

$$X(s) = \mathcal{L}[e^{At}]x(0) + \mathcal{L}[e^{At}]BU(s)$$

$$x(t) = \int_0^t e^{At}\, Bu(\tau)d\tau$$

$$\Rightarrow\ e^{At}$$

➔ State-Transition Matrix Φ

- $\Phi(t) = e^{At}\ = \mathcal{L}^{-1}[(sI - A)^{-1}]$
- It contains all the information about the free motions of the system described by $\dot{x} = Ax$

<u>LTI</u> Properties:

- $\Phi(0) = e^{0t} = I$
- $\Phi^{-1}(t) = \Phi(-t)$
- $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$
- $[\Phi(t)]^n = \Phi(nt)$

➔ The closed-loop poles are the eignvalues of the system matrix

Digital State Space:

- Difference equations in state-space form:

$$x[n+1] = Ax[n] + Bu[n]$$
$$y[n] = Cx[n] + Du[n]$$

- Where:
  - u[n], y[n]: input & output (scalars)
  - x[n]: state vector

Digital Control Law Design

In Chapter 2, we saw that the state-space description of a continuous system is given by (2.43),

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u, \tag{6.1}$$

and (2.44),

$$y = \mathbf{H}\mathbf{x}. \tag{6.2}$$

We assume the control is applied from the computer by a ZOH as shown in Fig. 1.1. Therefore, (6.1) and (6.2) have an exact discrete representation as given by (2.57),

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k),$$
$$y(k) = \mathbf{H}\mathbf{x}(k), \tag{6.3}$$

where

$$\boxed{\mathbf{\Phi} = e^{\mathbf{F}T},} \tag{6.4a}$$
$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G}, \tag{6.4b}$$

Discretisation (FPW!)

- We can use the time-domain representation to produce difference equations!

$$\boldsymbol{x}(kT+T) = e^{\mathbf{F}T}\ \boldsymbol{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

Notice $\boldsymbol{u}(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series.

We can structure this by using the form:

$$u(\tau) = u(kT), \qquad kT \leq \tau \leq kT+T$$

State-space z-transform

We can apply the z-transform to our system:

$$(z\mathbf{I} - \boldsymbol{\Phi})\boldsymbol{X}(z) = \boldsymbol{\Gamma}U(k)$$
$$Y(z) = \mathbf{H}\boldsymbol{X}(z)$$

which yields the transfer function:

$$\frac{Y(z)}{\boldsymbol{X}(z)} = G(z) = \mathbf{H}(z\mathbf{I} - \boldsymbol{\Phi})^{-\mathbf{1}}\boldsymbol{\Gamma}$$

State-space control design
¿¿¿Que pasa????

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:

$$u = -\mathbf{K}\boldsymbol{x}$$

such that $\det(z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K}) = \alpha_c(z)$

where $\alpha_c(z)$ is the desired control characteristic equation

Predictably, this requires the system controllability matrix $\boldsymbol{\mathcal{C}} = [\mathbf{\Gamma} \quad \mathbf{\Phi}\mathbf{\Gamma} \quad \mathbf{\Phi}^2\mathbf{\Gamma} \quad \cdots \quad \mathbf{\Phi}^{n-1}\mathbf{\Gamma}]$ to be full-rank.

Solving State Space

A Systematic Procedure for Determining State Eqs.

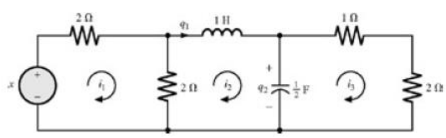
1. Choose all independent capacitor voltages and inductor currents to be the state variables.

2. Choose a set of loop currents; express the state variables and their first derivatives in terms of these loop currents.

3. Write the loop equations and eliminate all variables other than state variables (and their first derivatives) from the equations derived in Steps 2 and 3.

See also: Lathi § 13.2-1 (p. 788)

A Quick Example



1. The inductor current q_1 and the capacitor voltage $q2$ as the state variables.

2. $q_1 = i_2$
   $\frac{1}{2}\dot{q}_2 = i_2 - i_3$

3. $4i_1 - 2i_2 = x$
   $2(i_2 - i_1) + \dot{q}_1 + q_2 = 0$
   $-q_2 + 3i_3 = 0$

$$\dot{q}_1 = 2(i_1 - i_2) - q_2$$

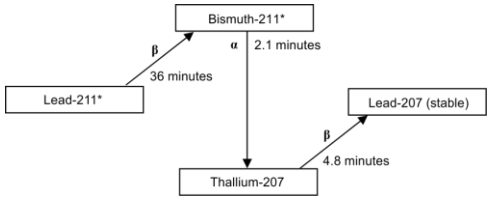
$$\dot{q}_1 = -q_1 - q_2 + \tfrac{1}{2}x$$

$$\dot{q}_2 = 2q_1 - \tfrac{2}{3}q_2$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} x$$

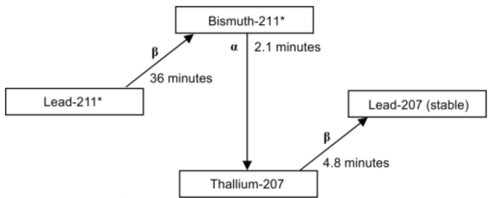
See also: Fig. 13.2, Lathi p. 789

Another Example



- $\frac{dN1(t)}{dt} = -\lambda_1 N1(t)$
- $\frac{dN2(t)}{dt} = -\lambda_2 N2(t) + \lambda_1 N1(t)$
- $\frac{dN3(t)}{dt} = -\lambda_3 N3(t) + \lambda_2 N2(t)$
- $\frac{dN4(t)}{dt} = \lambda_3 N3(t)$

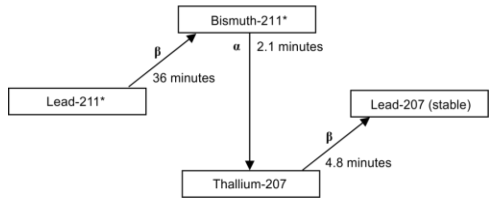
Another Example



$$X = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \rightarrow \dot{X} = \begin{bmatrix} \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}_3 \\ \dot{N}_4 \end{bmatrix}$$

$$\dot{X} = FX \rightarrow \begin{bmatrix} \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}_3 \\ \dot{N}_4 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 & 0 \\ 0 & \lambda_2 & -\lambda_3 & 0 \\ 0 & 0 & \lambda_3 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$$

Another Example



- $N_1(t)=N_1(0)\exp(-\lambda_1 t)$
- $N2(t) = N2(0)exp(-\lambda_2 t) - N1(0)\frac{\lambda_1}{\lambda_2-\lambda_1}(exp(-\lambda_2 t) - exp(-\lambda_1 t))$
- $N3(t) = \lambda_1\lambda_2 N1(0)\left[\frac{exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} + \frac{exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right]$
- $N4(t) = \lambda_1\lambda_2\lambda_3 N1(0)\left[\frac{exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(-\lambda_1)} + \frac{exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(-\lambda_2)} + \frac{exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(-\lambda_3)} + \frac{1}{(\lambda_1\lambda_2\lambda_3)}\right]$

Example: PID control

- Consider a system parameterised by three states:
  - x_1, x_2, x_3
  - where $x_2 = \dot{x}_1$ and $x_3 = \dot{x}_2$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}\boldsymbol{x} - \mathbf{K}u$$
$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\boldsymbol{x} + 0u$$

x_2 is the output state of the system;
x_1 is the value of the integral;
x_3 is the velocity.

Example: PID control [2]

- We can choose **K** to move the eigenvalues of the system as desired:

$$\det\begin{bmatrix} 1-K_1 & & \\ & 1-K_2 & \\ & & -2-K_3 \end{bmatrix} = \mathbf{0}$$

All of these eigenvalues must be positive.

It's straightforward to see how adding derivative gain K_3 can stabilise the system.

Can you use this for more than Control?

YES!

Frequency Response in State Space

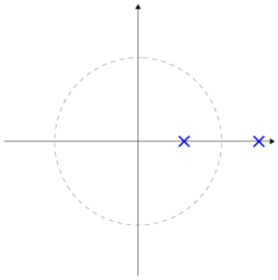
$$H(z) = C(zI - A)^{-1}B + D = \frac{1}{100z^2 - 200z + 80}$$

Poles at $\approx 0.55, 1.45$.

Eigenvalues of A:

$1, 1, 1.45, .55$

What are the (physical) implications?

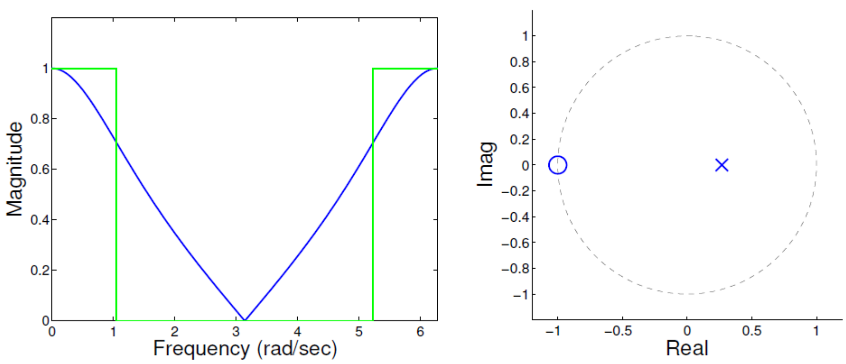


The Approach:

- Formulate the goal of control as an **optimization** (e.g. minimal impulse response, minimal effort, ...).
- You've already seen some examples of optimization-based design:
  - Used least-squares to obtain an FIR system which matched (in the least-squares sense) the desired frequency response.
  - Poles/zeros lecture: Butterworth filter

Discrete Time Butterworth Filters

"Maximally-flat filter". Sacrifice sharpness to have flat response in pass band and stop band.



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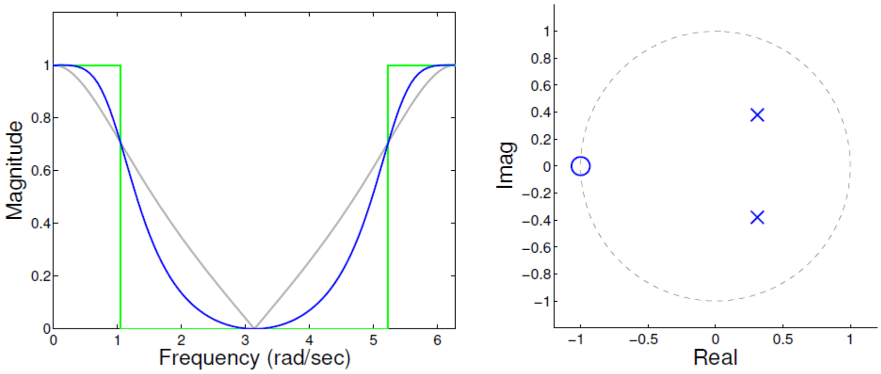

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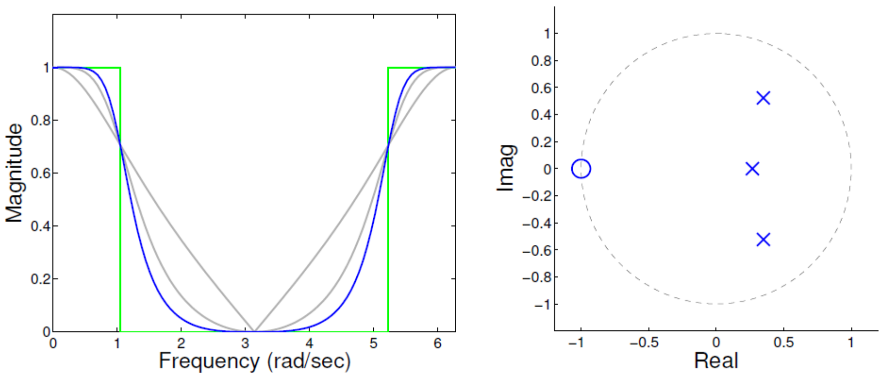

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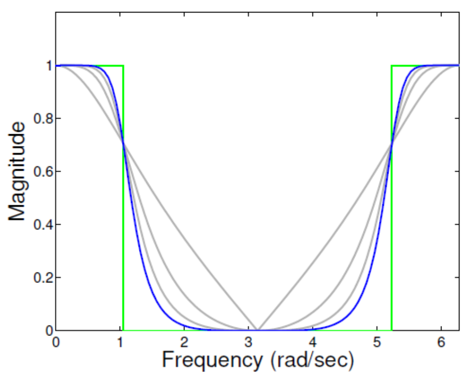

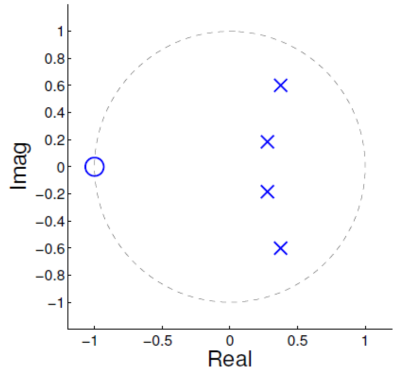

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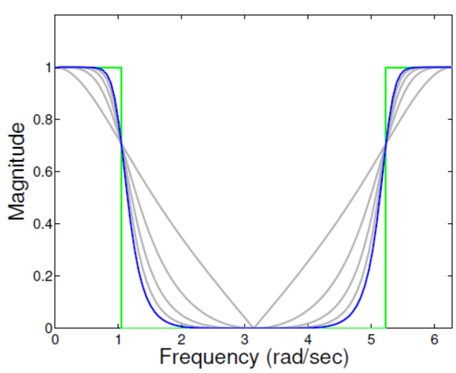

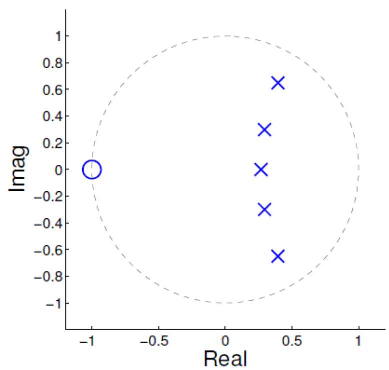

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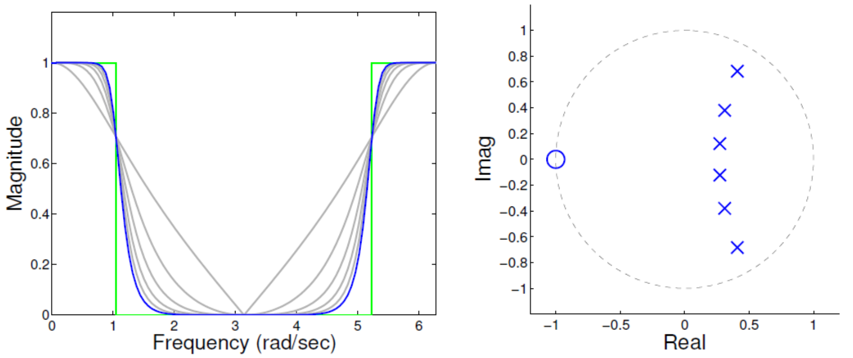

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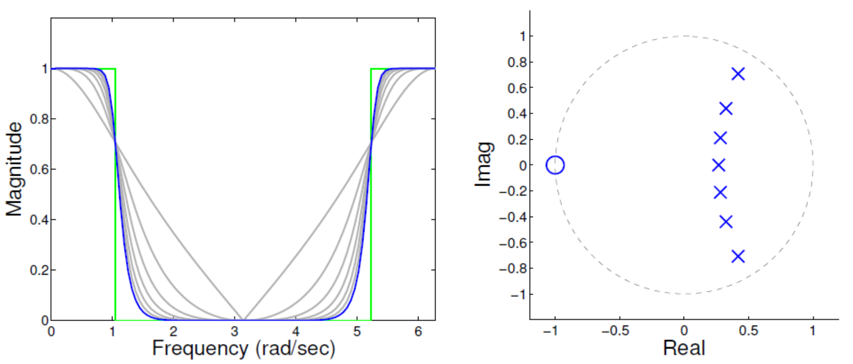


How?

- Constrained Least-Squares …

One formulation: Given $x[0]$

$$\underset{u[0],u[1],\dots,u[N]}{\text{minimize}} \quad ||\vec{u}||^2, \quad \text{where } \vec{u} = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N] \end{bmatrix}$$

$$\text{subject to} \quad x[N] = 0.$$

Note that

$$x[n] = A^n x[0] + \sum_{k=0}^{n-1} A^{(n-1-k)} B u[k],$$

so this problem can be written as

$$\underset{x_{ls}}{\text{minimize}} \, ||A_{ls} x_{ls} - b_{ls}||^2 \quad \text{subject to} \quad C_{ls} x_{ls} = D_{ls}.$$

Break ☺

Example 1:
`tf2ss`

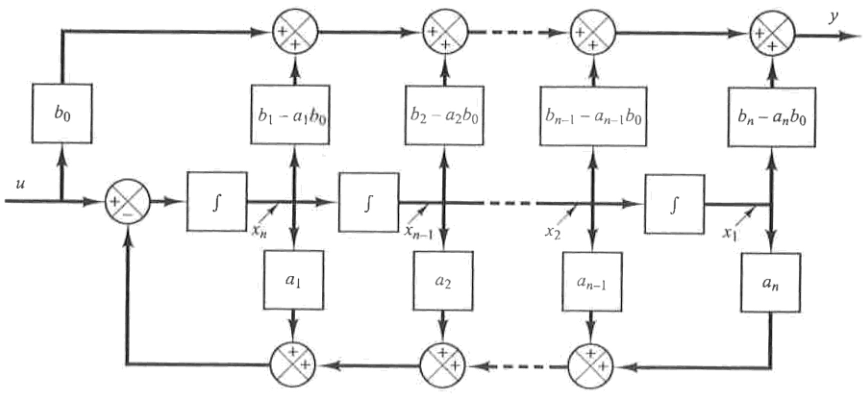
TF 2 SS – Control Canonical Form)

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Control Canonical Form as a Block Diagram



Modal Form

- CCF is not the only way to tf2ss

- Partial-fraction expansion of the system

➔ System poles appear as diagonals of Am

- Two issues:
  – The elements of matrix maybe complex if the poles are complex
  – It is non-diagonal with repeated poles

Modal Form

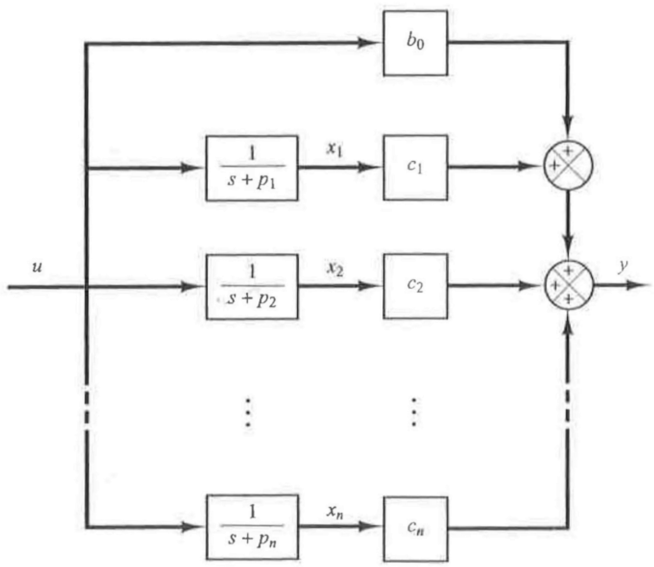
$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Modal Form Block Diagram



Matlab's tf2ss

- Given: $\frac{Y(s)}{U(s)} = \frac{25.04s+5.008}{s^3+5.03247s^2+25.1026s+5.008}$
  Get a state space representation of this system

- Matlab:

```
num = [25.04 5.008];
den = [1 5.03247 25.1026 5.008];
[A,B,C,D] = tf2ss(num/den);
```

- Answer:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5.0325 & -25.1026 & -5.008 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 25.04 \quad 5.008] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Example 2:
Obtaining a Time Response

From SS to Time Response — Impulse Functions

- Given: $\dot{x} = Ax + Bu$
- Solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Substituting $t_0 = 0$ into this:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Write the impulse as: $\mathbf{u}(t) = \delta(t)\mathbf{w}$
  - where **w** is a vector whose components are the magnitudes of **r** impulse functions applied at t=0

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)\mathbf{w}\,d\tau$$

$$= e^{\mathbf{A}t}\mathbf{x}(0-) + e^{\mathbf{A}t}\mathbf{B}\mathbf{w}$$

From SS to Time Response — Step Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = \boldsymbol{k}$
  Where **k** is a vector whose components are the magnitudes of *r* step functions applied at t=0.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_{0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{k}\,d\tau$$

$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[\int_{0}^{t}\left(\mathbf{I} - \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} - \cdots\right)d\tau\right]\mathbf{B}\mathbf{k}$$

$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{\mathbf{A}^2t^3}{3!} - \cdots\right)\mathbf{B}\mathbf{k}$$

  - Assume A is non-singular

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[-(\mathbf{A}^{-1})(e^{-\mathbf{A}t} - \mathbf{I})\right]\mathbf{B}\mathbf{k}$$

$$= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}\mathbf{k}$$

From SS to Time Response — Ramp Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = t\boldsymbol{v}$

  Where **v** is a vector whose components are magnitudes of ramp functions applied at $t = 0$

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\tau\mathbf{v}\,d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\tau\,d\tau\,\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\frac{\mathbf{I}}{2}t^2 - \frac{2\mathbf{A}}{3!}t^3 + \frac{3\mathbf{A}^2}{4!}t^4 - \frac{4\mathbf{A}^3}{5!}t^5 + \cdots\right)\mathbf{B}\mathbf{v}\end{aligned}$$

  - Assume A is non-singular

$$\Rightarrow \quad \begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + (\mathbf{A}^{-2})(e^{\mathbf{A}t} - \mathbf{I} - \mathbf{A}t)\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \left[\mathbf{A}^{-2}(e^{\mathbf{A}t} - \mathbf{I}) - \mathbf{A}^{-1}t\right]\mathbf{B}\mathbf{v}\end{aligned}$$

Example: Obtain the Step Response

- Given: $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_1\end{bmatrix} = \begin{bmatrix}-1 & -0.5 \\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0.5 \\ 0\end{bmatrix}u, \quad \begin{bmatrix}x_1(0) \\ x_2(0)\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$

$$y = [1 \quad 0]\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \qquad u(t) = 1(t)$$

- Solution:

$$\mathbf{A} = \begin{bmatrix}-1 & -0.5 \\ 1 & 0\end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix}0.5 \\ 0\end{bmatrix} \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$\begin{aligned}(s\mathbf{I} - \mathbf{A})^{-1} &= \begin{bmatrix}s+1 & 0.5 \\ -1 & s\end{bmatrix}^{-1} = \frac{1}{s^2 + s + 0.5}\begin{bmatrix}s & -0.5 \\ 1 & s+1\end{bmatrix} \\ &= \begin{bmatrix}\dfrac{s + 0.5 - 0.5}{(s+0.5)^2 + 0.5^2} & \dfrac{-0.5}{(s+0.5)^2 + 0.5^2} \\ \dfrac{1}{(s+0.5)^2 + 0.5^2} & \dfrac{s + 0.5 + 0.5}{(s+0.5)^2 + 0.5^2}\end{bmatrix}\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}\mathbf{\Phi}(t) = e^{\mathbf{A}t} &= \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] \\ &= \begin{bmatrix}e^{-0.5t}(\cos 0.5t - \sin 0.5t) & -e^{-0.5t}\sin 0.5t \\ 2e^{-0.5t}\sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t)\end{bmatrix}\end{aligned}$$

  - Set k=1, x(0)=0:

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}k \\ &= \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B} \\ &= \begin{bmatrix}0 & 1 \\ -2 & -2\end{bmatrix}\begin{bmatrix}0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t}\sin 0.5t\end{bmatrix} \\ &= \begin{bmatrix}e^{-0.5t}\sin 0.5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1\end{bmatrix}\end{aligned}$$

$$\Rightarrow \quad y(t) = [1 \quad 0]\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = x_1 = e^{-0.5t}\sin 0.5t$$

Example II: Obtain the Step Response

- Given:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad u(t) = 1(t)$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Phi(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$

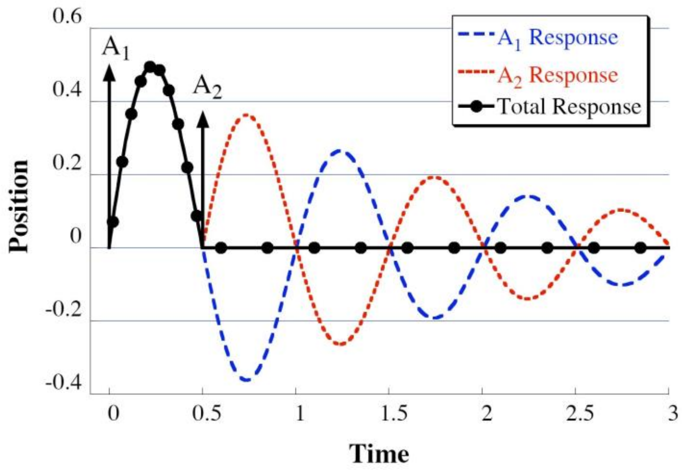
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

  – Assume x(0)=0:

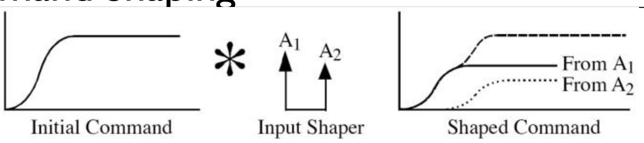
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Example 3: Command Shaping

Command Shaping



Command Shaping



- Zero Vibration (ZV)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{1+K} & \frac{K}{1+K} \\ 0 & \frac{T_d}{2} \end{bmatrix} \qquad K = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

- Zero Vibration and Derivative (ZVD)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{(1+K)^2} & \frac{2K}{(1+K)^2} & \frac{K^2}{(1+K)^2} \\ 0 & \frac{T_d}{2} & T_d \end{bmatrix}$$

Next Time…

- **Digital Feedback Control**

- Review:
  - Chapter 2 of FPW

- More Pondering??