PID & State-Space

http://elec3004.com

ELEC 3004: **Systems**: Signals & Controls
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Lecture 17

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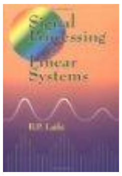
Lecture Schedule:

Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Systems: Linear Differential Systems
3	14-Mar	Sampling Theory & Data Acquisition
	16-Mar	Aliasing & Antialiasing
4	21-Mar	Discrete Time Analysis & Z-Transform
	23-Mar	Second Order LTID (& Convolution Review)
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
6	4-Apr	Digital Filters (IIR) & Filter Analysis
	6-Apr	Digital Filter (FIR)
7	11-Apr	Digital Windows
	13-Apr	FFT
	18-Apr	Holiday
	20-Apr	
	25-Apr	
8	27-Apr	Active Filters & Estimation
9	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	PID & State-Space
	11-May	State-Space Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response
	25-May	Applications in Industry
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

G. Franklin,
J. Powell,
M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990
[Available as UQ Ebook]

Today

➔ **P - I - D**

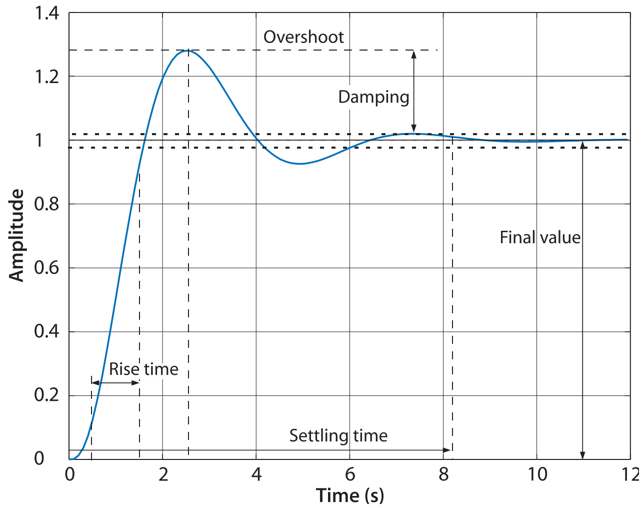
- FPW
  – Chapter 4:
    Discrete Equivalents to Continuous Transfer Functions: The Digital Filter

- FPW
  – Chapter 5: Design of Digital Control Systems Using Transform Techniques

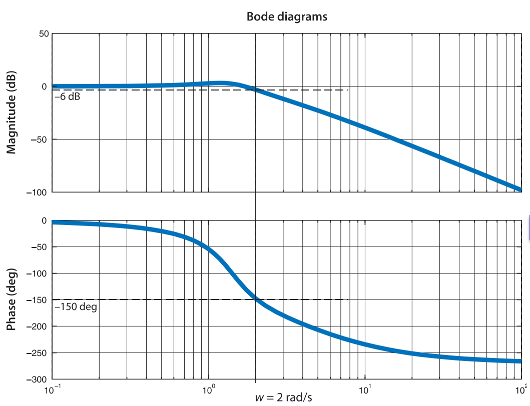
Next Time

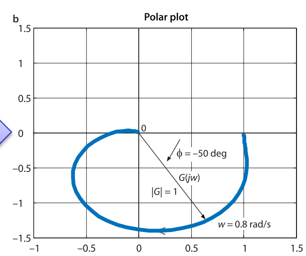
Feedback as a Filter

Time Response



Frequency Domain Analysis

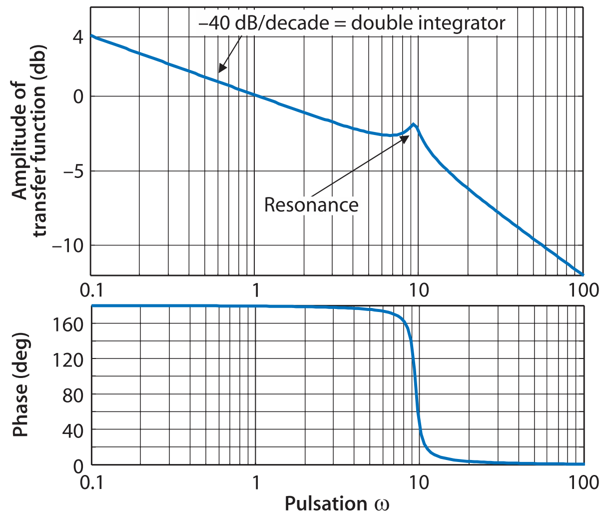




- Bode (Magnitude + Phase Plots)
- Nyquist Plot (Polar)

In This Way Feedback May Be Seen as a Filter

- Ex: Lightly Damped Robot Arm

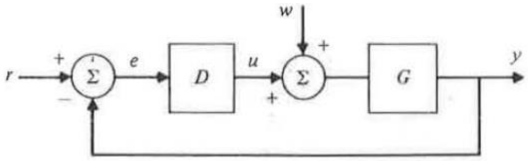


PID
(Intro)

PID

- Three basic types of control:
  - Proportional
  - Integral, and
  - Derivative

- The next step up from lead compensation
  - Essentially a combination of proportional and derivative control



Proportional Control

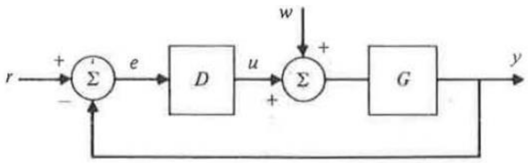
A discrete implementation of proportional control is identical to continuous; that is, where the continuous is

$$u(t) = K_p e(t) \quad \Rightarrow \quad D(s) = K_p,$$

the discrete is

$$u(k) = K_p e(k) \quad \Rightarrow \quad \boxed{D(z) = K_p}$$

where $e(t)$ is the error signal as shown in Fig 5.2.



PID Control

$$\boxed{D(z) = K_p\left(1 + \frac{Tz}{T_I(z-1)} + \frac{T_D(z-1)}{Tz}\right).}$$

The user simply has to determine the best values of

- K_p
- T_D and
- T_I

Another way to see P I|D

- Derivative

  D provides:

  – High sensitivity

  – Responds to change

  – Adds "damping" & ∴ permits larger K_P

  – Noise sensitive

  – Not used alone (∵ its on rate change of error – by itself it wouldn't get there)

→ "**D**iet Coke of control"

- Integral

  – Eliminates offsets (makes regulation ☺)

  – Leads to Oscillatory behaviour

  – Adds an "order" but instability (Makes a 2nd order system 3rd order)

→ "**I**nteresting cake of control"

Integral

- Integral applies control action based on accumulated output error
  - Almost always found with P control
- Increase dynamic order of signal tracking
  - Step disturbance steady-state error goes to zero
  - Ramp disturbance steady-state error goes to a constant offset

Let's try it!

Integral Control

For continuous systems, we integrate the error to arrive at the control,

$$u(t) = \frac{K_p}{T_I} \int_{t_o}^{t} e(t)dt \;\Rightarrow\; D(s) = \frac{K_p}{T_I s},$$

where T_I is called the *integral*, or *reset time*. The discrete equivalent is to sum all previous errors, yielding

$$u(k) = u(k-1) + \frac{K_p T}{T_I} e(k) \;\Rightarrow\; \boxed{D(z) = \frac{K_p T}{T_I(1 - z^{-1})} = \frac{K_p T z}{T_I(z-1)}.} \quad (5.60)$$

Just as for continuous systems, the primary reason for integral control is to reduce or eliminate steady-state errors, but this typically occurs at the cost of reduced stability.

Integral: P Control only

- Consider a first order system with a constant load disturbance, w; (recall as $t \rightarrow \infty$, $s \rightarrow 0$)

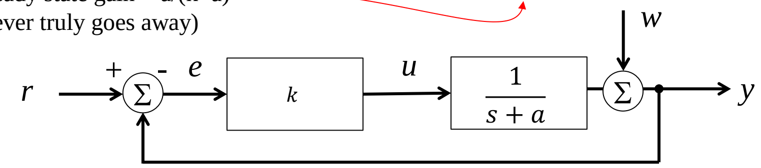
$$y = k\frac{1}{s+a}(r-y) + w$$

$$(s+a)y = k\,(r-y) + (s+a)w$$

$$(s+k+a)y = kr + (s+a)w$$

$$y = \frac{k}{s+k+a}r + \underbrace{\frac{(s+a)}{s+k+a}}w$$

Steady state gain = a/(k+a)
(never truly goes away)



Now with added integral action

$$y = k\left(1 + \frac{1}{\tau_i s}\right)\frac{1}{s+a}(r-y) + w$$

$$y = k\frac{s+\tau_i^{-1}}{s}\boxed{\frac{1}{s+a}}(r-y) + w$$

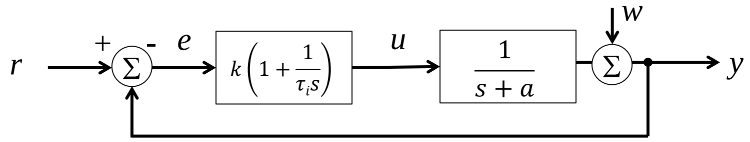
Same dynamics

$$s(s+a)y = k(s+\tau_i^{-1})(r-y) + s(s+a)w$$

$$(s^2 + (k+a)s + \tau_i^{-1})y = k(s+\tau_i^{-1})r + s(s+a)w$$

$$y = \frac{k(s+\tau_i^{-1})}{(s^2+(k+a)s+\tau_i^{-1})}r + \frac{\boxed{s(s+a)}}{k(s+\tau_i^{-1})}w$$

Must go to zero
for constant *w*!



Derivative Control

For continuous systems, derivative or rate control has the form

$$u(t) = K_p T_D \dot{e}(t) \quad \Rightarrow \quad D(s) = K_p T_D s$$

where T_D is called the *derivative time*. Differentiation can be approximated in the discrete domain as the first difference, that is,

$$u(k) = K_p T_D \frac{(e(k) - e(k-1))}{T} \quad \Rightarrow \quad \boxed{D(z) = K_p T_D \frac{1 - z^{-1}}{T} = K_p T_D \frac{z-1}{Tz.}}$$

In many designs, the compensation is a sum of proportional and derivative control (or PD control). In this case, we have

$$D(z) = K_p \left(1 + \frac{T_D(z-1)}{Tz}\right).$$

or, equivalently,

$$\boxed{D(z) = K\frac{z-\alpha}{z}}$$

Derivative Control [2]

- Similar to the lead compensators
  - The difference is that the pole is at $z = 0$

    [Whereas the pole has been placed at various locations along the z-plane real axis for the previous designs.]

- In the continuous case:
  - pure derivative control represents the ideal situation in that there is no destabilizing phase lag from the differentiation
  - the pole is at $s = -\infty$

- In the discrete case:
  - z=0
  - However this has phase lag because of the necessity to wait for one cycle in order to compute the first difference

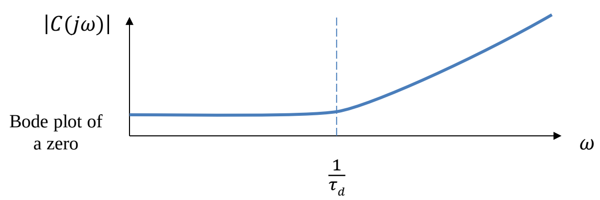
Derivative

- Derivative uses the rate of change of the error signal to anticipate control action
  - Increases system damping (when done right)
  - Can be thought of as ‘leading’ the output error, applying correction predictively
  - Almost always found with P control*

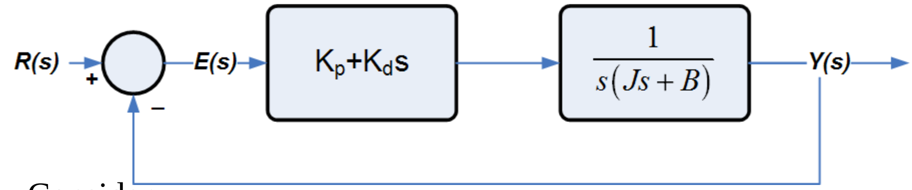
**What kind of system do you have if you use D, but don’t care about position?* Is it the same as P control in velocity space?

Derivative

- It is easy to see that PD control simply adds a zero at $s = -\frac{1}{\tau_d}$ with expected results
  - Decreases dynamic order of the system by 1
  - Absorbs a pole as $k \rightarrow \infty$
- Not all roses, though: derivative operators are sensitive to high-frequency noise



PD for 2nd Order Systems



- Consider:

$$\frac{Y(s)}{R(s)} = \frac{(K_P + K_D s)}{Js^2 + (B + K_D)s + K_P}$$

- Steady-state error: $e_{ss} = \frac{B}{K_P}$
- Characteristic equation: $Js^2 + (B + K_D)s + K_P = 0$
- Damping Ratio: $\zeta = \frac{B+K_D}{2\sqrt{K_P J}}$

➔It is possible to make e_{ss} and overshoot small (↓) by making B small (↓), K_P large ↑, K_D such that ζ:between [0.4 – 0.7]

PID – Control for the PID-dly minded

- Proportional-Integral-Derivative control is the control engineer's hammer*
  - For P,PI,PD, etc. just remove one or more terms

$$\mathrm{C(s)} = k\left(1 + \frac{1}{\tau i s} + \tau d s\right)$$

Proportional
Integral
Derivative

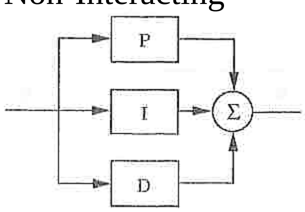
*Everything is a nail. That's why it's called "Bang-Bang" Control ☺

PID

- Collectively, PID provides two zeros plus a pole at the origin
  - Zeros provide phase lead
  - Pole provides steady-state tracking
  - Easy to implement in microprocessors
- Many tools exist for optimally tuning PID
  - Zeigler-Nichols
  - Cohen-Coon
  - Automatic software processes

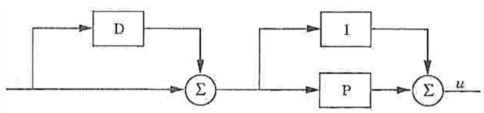
PID Implementation

- Non-Interacting



$$C(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)$$

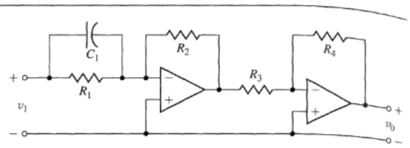
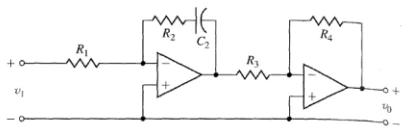
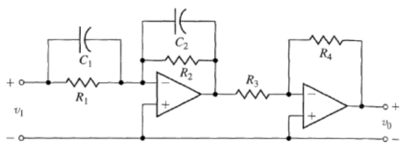
- Interacting Form



$$C'(s) = K\left(1 + \frac{1}{sT_i}\right)(1 + sT_d)$$

- Note: Different K,T_i and T_d

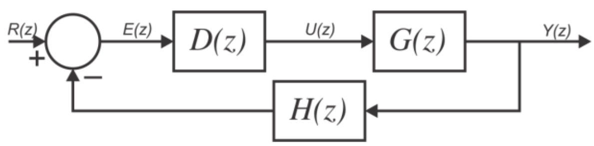
Operational Amplifier Circuits for Compensators

Type of Controller	$G_c(s) = \frac{V_0(s)}{V_1(s)}$	
PD	$G_c = \frac{R_4R_2}{R_3R_1}(R_1C_1s + 1)$	
PI	$G_c = \frac{R_4R_2(R_2C_2s + 1)}{R_3R_1(R_2C_2s)}$	
Lead or lag Lead if $R_1C_1 > R_2C_2$ Lag if $R_1C_1 < R_2C_2$	$G_c = \frac{R_4R_2(R_1C_1s + 1)}{R_3R_1(R_2C_2s + 1)}$	

- (Yet Another Way to See PID)

Source: Dorf & Bishop, *Modern Control Systems*, p. 828

PID as Difference Equation



$$\frac{U(z)}{E(z)} = D(z) = K_p + K_i\left(\frac{Tz}{z-1}\right) + K_d\left(\frac{z-1}{Tz}\right)$$

$$\boxed{u(k) = \left[K_p + K_iT + \left(\frac{K_d}{T}\right)\right] \cdot e(k) - [K_dT] \cdot e(k-1) + [K_i] \cdot u(k-1)}$$

PID Algorithm (in various domains):

FPW § 5.8.4 [p.224]

- PID Algorithm (in Z-Domain):

$$D(z) = K_p\left(1 + \frac{Tz}{T_I(z-1)} + \frac{T_D(z-1)}{Tz}\right)$$

- As Difference equation:

$$u(t_k) = u(t_{k-1}) + K_p\left[\left(1 + \frac{\Delta t}{T_i} + \frac{T_d}{\Delta t}\right)e(t_k) + \left(-1 - \frac{2T_d}{\Delta t}\right)e(t_{k-1}) + \frac{T_d}{\Delta t}e(t_{k-2})\right]$$

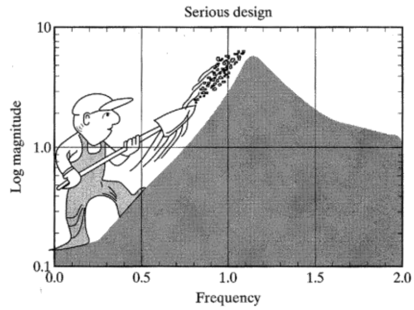
- Pseudocode [Source: Wikipedia]:

```
previous_error = 0, integral = 0
start:
  error = setpoint - measured_value
  integral = integral + error*dt
  derivative = (error - previous_error)/dt
  output = Kp*error + Ki*integral + Kd*derivative
  previous_error = error
  wait(dt)
  goto start
```


Break ☺

Seeing PID – No Free Lunch

- The energy (and sensitivity) moves around (in this case in “frequency”)



- Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.

Source: Gunter Stein's interpretation of the water bed effect – G. Stein, *IEEE Control Systems Magazine*, 2003.

When Can PID Control Be Used?

When:

- “Industrial processes” such that the demands on the performance of the control are not too high.
  - Control authority/actuation
  - Fast (clean) sensing
- PI: Most common
  - All stable processes can be controlled by a PI law (modest performance)
  - First order dynamics

PID (PI + Derivative):

- Second order (A double integrator cannot be controlled by PI)
- Speed up response When time constants differ in magnitude (Thermal Systems)

Something More Sophisticated:

- Large time delays
- Oscillatory modes between inertia and compliances

PID Intuition

$$u(t) = K\left[e(t) + \frac{1}{T_i}\int e(s)\,ds + T_d\frac{de(t)}{dt}\right]$$

- P:
  - Control action is proportional to control error
  - It is necessary to have an error to have a non-zero control signal

- I:
  - The main function of the integral action is to make sure that the process output agrees with the set point in steady state

PID Intuition

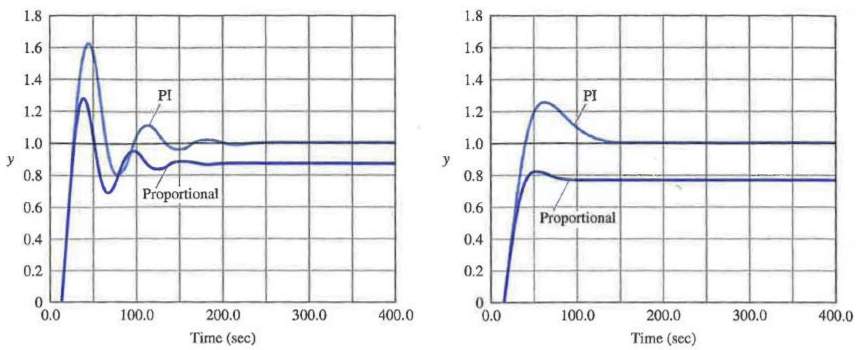
$$u(t) = K\left[e(t) + \frac{1}{T_i}\int e(s)\,ds + T_d\frac{de(t)}{dt}\right]$$

- P:
- I:
- D:
  - The purpose of the derivative action is to improve the closed loop stability.
  - The instability "mechanism" "controlled" here is that because of the process dynamics it will take some time before a change in the control variable is noticeable in the process output.
  - The action of a controller with proportional and derivative action may e interpreted as if the control is made proportional to the *predicted* process output, where the prediction is made by extrapolating the error by the tangent to the error curve.

PID Intuition

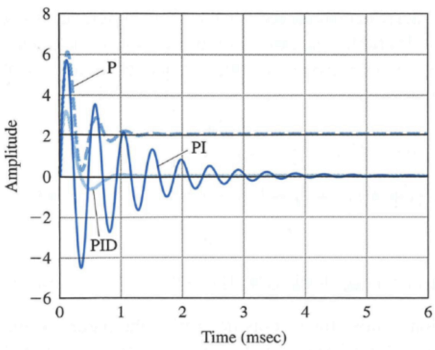
Effects of increasing a parameter independently					
Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability
K_p	↓	⇑	Minimal change	↓	↓
K_I	↓	⇑	⇑	Eliminate	↓
K_D	Minor change	↓	↓	No effect / minimal change	Improve (if K_D small)

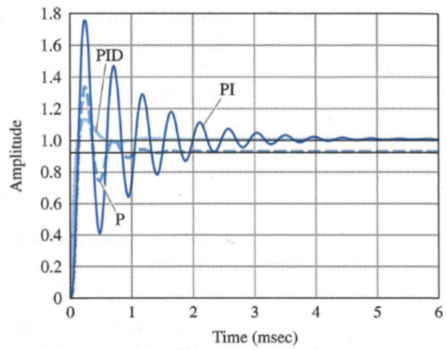
PID Intuition: P and PI



PID Intuition: P and PI and PID

- Responses of P, PI, and PID control to





(a) step disturbance input

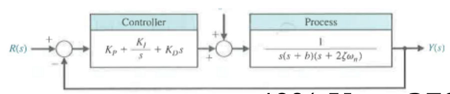
(b) step reference input

PID Example

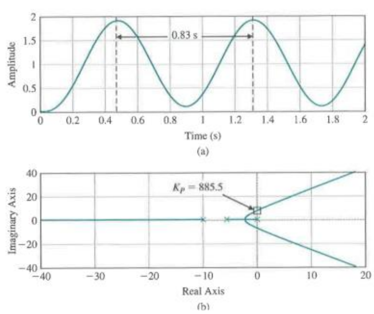
- A 3rd order plant: b=10, ζ=0.707, ω_n=4

$$G(s) = \frac{1}{s(s+b)(s+2\zeta\omega_n)}$$

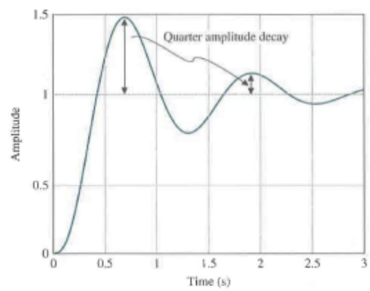
- PID:



- Kp=855:



- 40% Kp = 370



PID Tuning

PID Intuition & Tuning

- Tuning – How to get the "magic" values:
  - Dominant Pole Design
  - Ziegler Nichols Methods
  - Pole Placement
  - Auto Tuning

- Although PID is common it is often poorly tuned
  - The derivative action is frequently switched off!
  (Why ∵ it's sensitive to noise)
  - Also lots of "I" will make the system more transitory & leads to integrator wind-up.

Ziegler-Nichols Tuning – Reaction Rate

FPW § 5.8.5 [p.224]

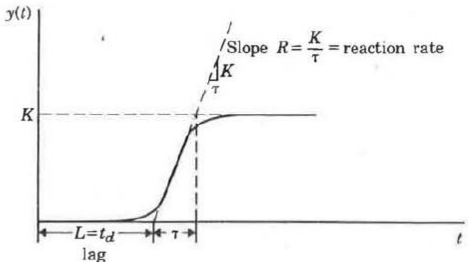
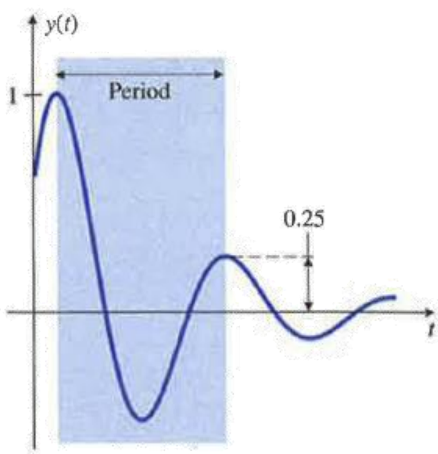


Table 5.2 Ziegler-Nichols tuning parameters using transient response.

	K_p	T_I	T_D
P	$1/RL$		
PI	$0.9/RL$	$3L$	
PID	$1.2/RL$	$2L$	$0.5L$

Quarter decay ratio



Ziegler-Nichols Tuning – Stability Limit Method

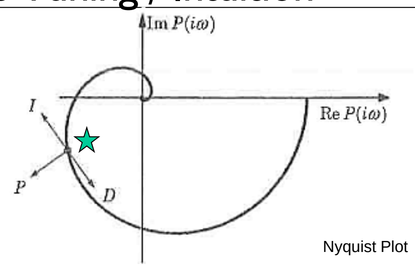
FPW § 5.8.5 [p.226]

- Increase K_P until the system has continuous oscillations
  - $\equiv K_U$: Oscillation Gain for "Ultimate stability"
  - $\equiv P_U$: Oscillation Period for "Ultimate stability"

Table 5.3 Ziegler-Nichols tuning parameters using stability limit.

	K_p	T_I	T_D
P	$0.5K_u$		
PI	$0.45K_u$	$P_u/1.2$	
PID	$0.6K_u$	$P_u/2$	$P_u/8$

Ziegler-Nichols Tuning / Intuition



$$C(i\omega_u) = K\left(1 + i\left(\omega_u T_d - \frac{1}{\omega_u T_i}\right)\right) \approx 0.6K_u(1 + 0.467i)$$

- For a Given Point (★), the effect of increasing P,I and D in the "s-plane" are shown by the arrows above Nyquist plot

Extension!:

2nd Order Responses

Review: Direct Design: Second Order Digital Systems

Consider the z-transform of a decaying exponential signal:

$y(t) = e^{-at}\cos(bt)\,\mathcal{U}(t)$ ($\mathcal{U}(t) =$ unit step)

- sample: $y(kT) = r^k \cos(k\theta)\,\mathcal{U}(kT)$ with $r = e^{-aT}$ & $\theta = bT$
- transform: $Y(z) = \frac{1}{2}\frac{z}{(z - re^{j\theta})} + \frac{1}{2}\frac{z}{(z - re^{-j\theta})}$

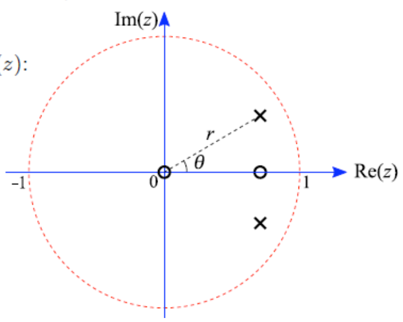
$$= \frac{z(z - r\cos\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

- e.g. y_k is the pulse response of $G(z)$:

$$G(z) = \frac{z(z - r\cos\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

poles: $\begin{cases} z = re^{j\theta} \\ z = re^{-j\theta} \end{cases}$

zeros: $\begin{cases} z = 0 \\ z = r\cos\theta \end{cases}$



Response of 2nd order system [1/3]

Responses for varying r:

- $r < 1$

  $\Downarrow$

  exponentially decaying envelope

- $r = 1$

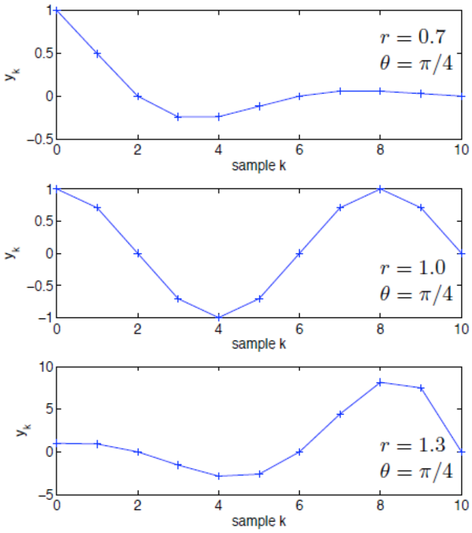
  $\Downarrow$

  sinusoidal response with $2\pi/\theta$ samples per period

- $r > 1$

  $\Downarrow$

  exponentially increasing envelope



Response of 2nd order system [2/3]

Responses for varying θ:

- $\theta = 0$

  $\Downarrow$

  decaying exponential

- $\theta = \pi/2$

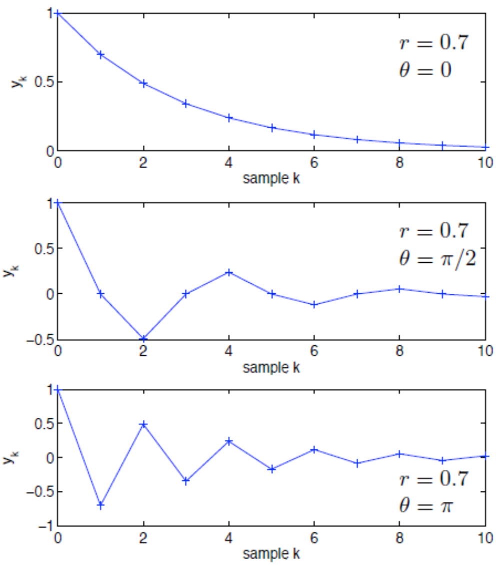
  $\Downarrow$

  $2\pi/\theta = 4$ samples per period

- $\theta = \pi$

  $\Downarrow$

  2 samples per period



Response of 2nd order system [3/3]

Some special cases:

- for $\theta = 0$, $Y(z)$ simplifies to:

$$Y(z) = \frac{z}{z - r}$$

$\Longrightarrow$ exponentially decaying response

- when $\theta = 0$ and $r = 1$:

$$Y(z) = \frac{z}{z - 1}$$

$\Longrightarrow$ unit step

- when $r = 0$:

$$Y(z) = 1$$

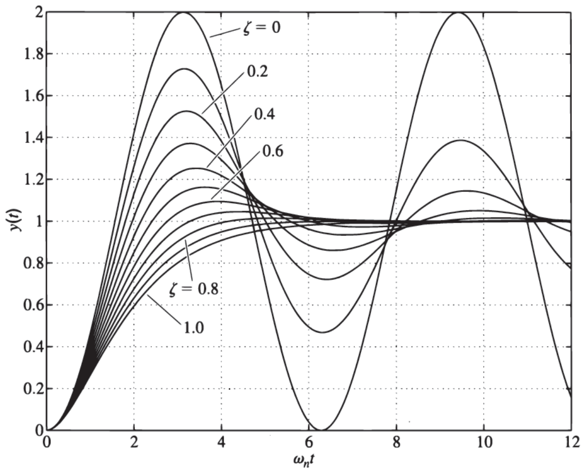
$\Longrightarrow$ unit pulse

- when $\theta = 0$ and $-1 < r < 0$:

samples of alternating signs

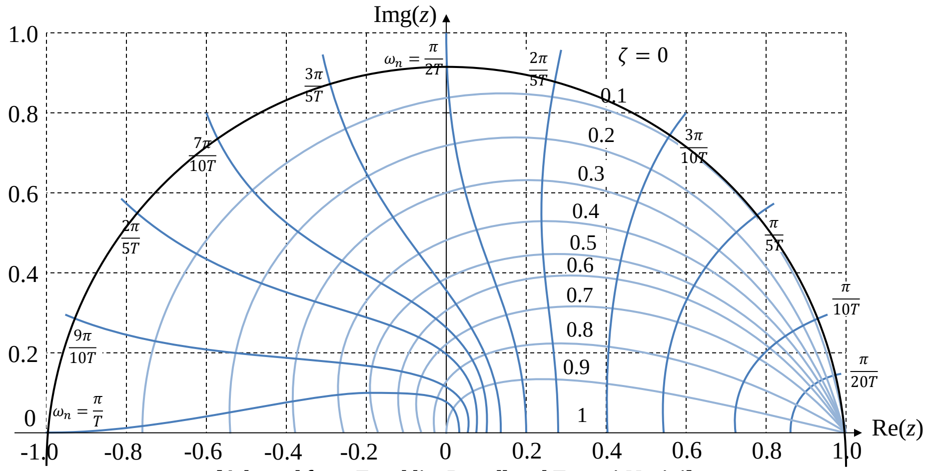
2nd Order System Response

- Response of a 2nd order system to increasing levels of damping:



Damping and natural frequency

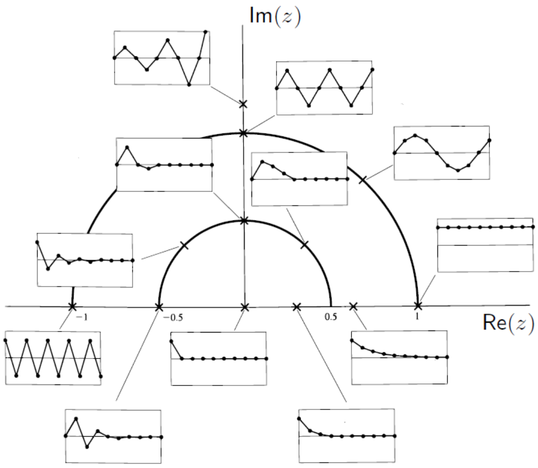
$$z = e^{sT} \text{ where } s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



[Adapted from Franklin, Powell and Emami-Naeini]

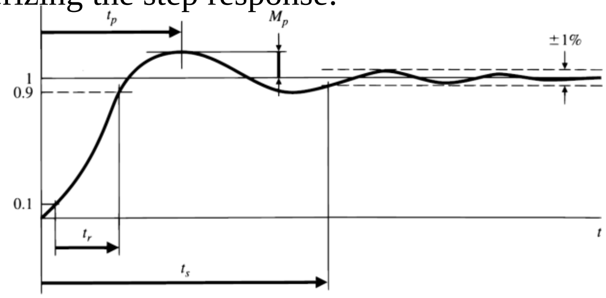
Pole positions in the z-plane

- Poles inside the unit circle are **stable**
- Poles outside the unit circle **unstable**
- Poles on the unit circle are oscillatory
- Real poles at 0 < z < 1 give exponential response
- Higher frequency of oscillation for larger
- Lower apparent damping for larer and r



2^{nd} Order System Specifications

Characterizing the step response:



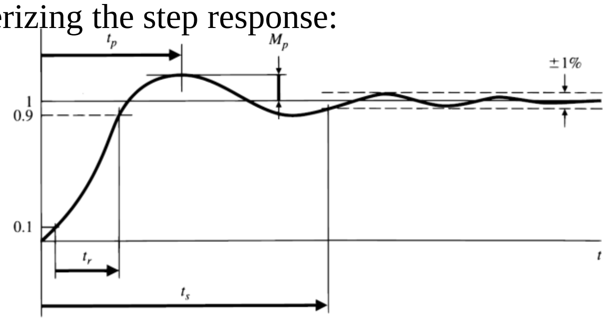
- Rise time (10% → 90%): $t_r \approx \frac{1.8}{\omega_0}$
- Overshoot: $M_p \approx \frac{e^{-\pi\zeta}}{\sqrt{1-\zeta^2}}$
- Settling time (**to 1%**): $t_s = \frac{4.6}{\zeta\omega_0}$
- Steady state error to unit step: e_{ss}
- Phase margin:

$$\phi_{PM} \approx 100\zeta$$

Why 4.6? It's **-ln(1%)**
$\rightarrow e^{-\zeta\omega_0} = 0.01 \rightarrow \zeta\omega_0 = 4.6 \rightarrow t_s = \frac{4.6}{\zeta\omega_0}$

2^{nd} Order System Specifications

Characterizing the step response:



- Rise time (10% → 90%) & Overshoot:
  t_r, M_p → ζ, ω_0 : Locations of dominant poles
- Settling time (to 1%):
  t_s → radius of poles: $|z| < 0.01^{\frac{T}{t_s}}$
- Steady state error to unit step:
  e_{ss} → final value theorem $e_{ss} = \lim_{z \to 1} \{(z-1)\,F(z)\}$

Ex: System Specifications → Control Design [1/4]

Design a controller for a system with:

- A continuous transfer function: $G(s) = \dfrac{0.1}{s(s+0.1)}$
- A discrete ZOH sampler
- Sampling time (T_s): T_s= 1s
- Controller:

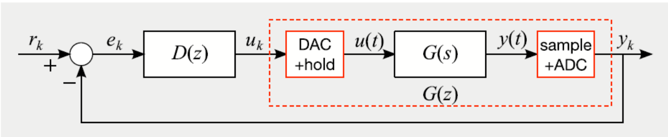
$$u_k = -0.5u_{k-1} + 13\,(e_k - 0.88e_{k-1})$$

The closed loop system is required to have:

- $M_p < 16\%$
- $t_s < 10$ s
- $e_{ss} < 1$

Ex: System Specifications → Control Design [2/4]

1. (a) Find the pulse transfer function of $G(s)$ plus the ZOH



$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{(z-1)}{z}\mathcal{Z}\left\{\frac{0.1}{s^2(s+0.1)}\right\}$$

e.g. look up $\mathcal{Z}\{a/s^2(s+a)\}$ in tables:

$$G(z) = \frac{(z-1)}{z}\,\frac{z\Big((0.1 - 1 + e^{-0.1})z + (1 - e^{-0.1} - 0.1e^{-0.1})\Big)}{0.1(z-1)^2(z-e^{-0.1})}$$

$$= \frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)}$$

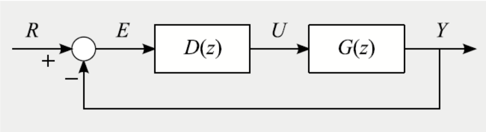
(b) Find the controller transfer function (using z = shift operator):

$$\frac{U(z)}{E(z)} = D(z) = 13\frac{(1-0.88z^{-1})}{(1+0.5z^{-1})} = 13\frac{(z-0.88)}{(z+0.5)}$$

Ex: System Specifications → Control Design [3/4]

2. Check the steady state error e_{ss} when $r_k =$ unit ramp

$$e_{ss} = \lim_{k\to\infty} e_k = \lim_{z\to 1}(z-1)E(z)$$



$$\frac{E(z)}{R(z)} = \frac{1}{1+D(z)G(z)}$$

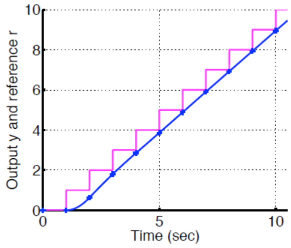
$$R(z) = \frac{Tz}{(z-1)^2}$$

so $$e_{ss} = \lim_{z\to 1}\left\{(z-1)\frac{Tz}{(z-1)^2}\frac{1}{1+D(z)G(z)}\right\} = \lim_{z\to 1}\frac{T}{(z-1)D(z)G(z)}$$

$$= \lim_{z\to 1}\frac{T}{(z-1)\dfrac{0.0484(z+0.9672)}{(z-1)(z-0.9048)}D(1)}$$

$$= \frac{1-0.9048}{0.0484(1+0.9672)D(1)} = 0.96$$

$\Longrightarrow$ $e_{ss} < 1$ (as required)



Ex: System Specifications → Control Design [4/4]

3. Step response: overshoot $M_p < 16\%$ $\Longrightarrow$ $\zeta > 0.5$

settling time $t_s < 10$ $\Longrightarrow$ $|z| < 0.01^{1/10} = 0.63$

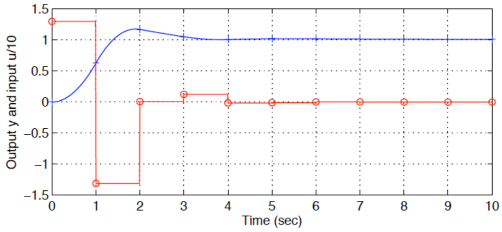
The closed loop poles are the roots of $1 + D(z)G(z) = 0$, i.e.

$$1 + 13\frac{(z-0.88)}{(z+0.5)}\frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)} = 0$$

$$\Longrightarrow z = 0.88,\ -0.050 \pm j0.304$$

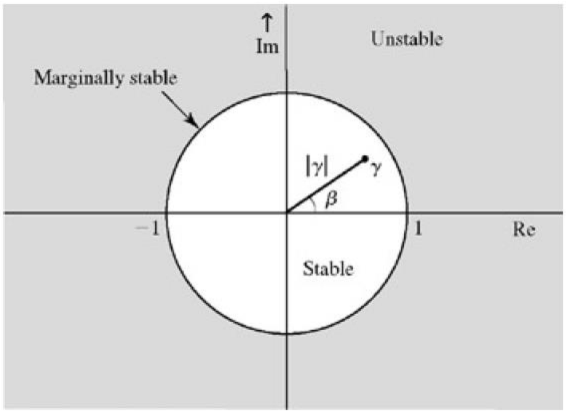
But the pole at $z = 0.88$ is cancelled by controller zero at $z = 0.88$, and

$$z = -0.050 \pm j0.304 = re^{\pm j\theta} \quad \Longrightarrow \begin{cases} r = 0.31,\ \theta = 1.73 \\ \zeta = 0.56 \end{cases}$$

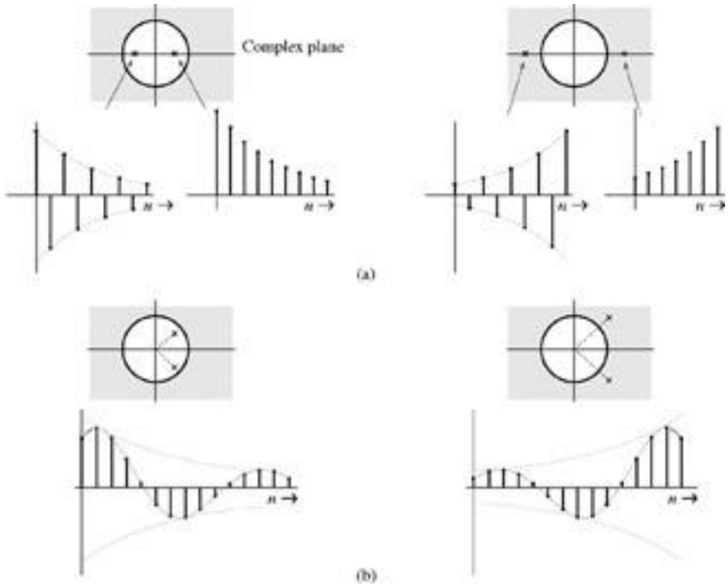


all specs satisfied!

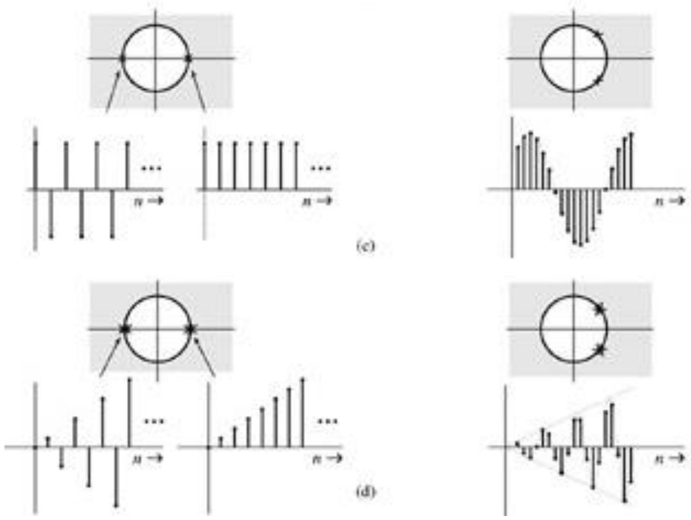
LTID Stability



Characteristic roots location
and the corresponding characteristic modes [1/2]



Characteristic roots location and the corresponding characteristic modes [2/2]



Next Time…

- **Digital Feedback Control**

- Review:
  - Chapter 2 of FPW

- More Pondering??