

Linearity:

Linearity is a powerful property for a system to have as it allows the use of several useful techniques for the analysis and design of systems. Therefore it is important to be able to recognise and test for linearity.

In order for a system to be linear it has satisfy the general condition that given that time varying input $x_1(t)$ produces time varying output $y_1(t)$ and input $x_2(t)$ produces output $y_2(t)$ then input $(ax_1(t) + bx_2(t))$ must produce output $ay_1(t) + by_2(t)$ for all scalar values of a , b and t (https://en.wikipedia.org/wiki/Linear_system) or in other terms,

Given that:

$$F(x_1(t)) = y_1(t)$$

and

$$F(x_2(t)) = y_2(t)$$

Then the system f is linear if

$$F(ax_1(t) + bx_2(t)) = ay_1(t) + by_2(t)$$

This property is often referred to as superposition and is tested by testing that the system has the properties of scaling (aka homogeneity) and additivity.

Additivity is the property that given

$$F(x_1(t)) = y_1(t) \quad \text{and} \quad F(x_2(t)) = y_2(t)$$

Then the system f is additive if

$$F(x_1(t) + x_2(t)) = y_1(t) + y_2(t)$$

For all t

Scaling is the property that given

$$F(x_1(t)) = y_1(t)$$

Then the system F is homogeneous if

$$F(ax_1(t)) = ay_1(t)$$

For all t and a .

Exercise: Prove each of the following is linear/non linear

- $f(x(t)) = x(t)^2$
- $f(x(t)) = 5x(t) - 3x(t - 1)$
- $f(x(t)) = (x(t)) \frac{dx}{dt}$

Causality Revision:

Causality is a property of systems that requires that the output at no point is dependent of future values. To test a system F for causality you must prove that no value $y(t)$ is dependent on any input $x(n)$ or $y(n)$ such that $n > t$.

The most common examples of non-causal systems are systems that have access to all input data at the start of processing and are therefore not “real time”.

Examples of causal systems:

$$y(t) = 4x(t)$$

$$y[n] = x[n - 1] \times x[n - 2]$$

$$y(t) = 4y(t - 1) + 7x(t)$$

Examples of non-causal systems:

$$y(t) = y(t + 1) + x(t)$$

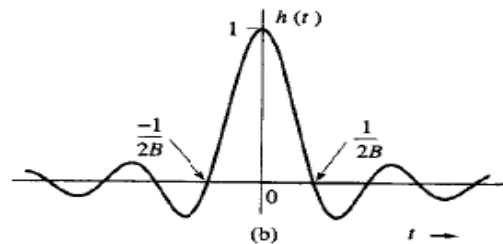
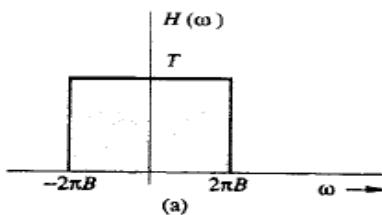
$$y(t) = \frac{x(t + 1)}{5}$$

Sinc Revision:

The sinc function is an important function in signal analysis. It is defined as

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

The useful thing about the sinc function is that it's Fourier transform is the rect function so convolution with the sinc function is an ideal low pass filter.



Lathi [p 325]

Basis Revision:

A basis is a set of n dimensional vectors that can be used to describe any n dimensional vector. Basis are most useful when they are orthogonal, meaning that each basis vector is perpendicular with all other vectors in the basis.

When $n = 3$ then one set of orthogonal basis vectors are $\mathbf{i} = [1 \ 0 \ 0]$, $\mathbf{j} = [0 \ 1 \ 0]$ and $\mathbf{k} = [0 \ 0 \ 1]$. Any vector in 3 dimensional space can be expressed as $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. You can also combine the basis vector into a matrix. From this we can say that any vector can be expressed as

$$\mathbf{V} = (a \ b \ c) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

However there are alternative sets of basis vectors for 3 dimensional space. Another set of basis vectors is $\mathbf{i} = [-1 \ 1 \ 0]$, $\mathbf{j} = [1 \ 1 \ 0]$ and $\mathbf{k} = [0 \ 0 \ 1]$. And therefore we can describe any 3 dimensional vector as

$$\mathbf{V} = (d \ e \ f) \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If you want to convert one basis to another then it can be done using matrix manipulation

Find d , e and f in relation to a , b and c

$$(d \ e \ f) \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (a \ b \ c) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(d \ e \ f) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (a \ b \ c) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$(d \ e \ f) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} (a \ b \ c) \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\therefore d = \frac{b - a}{2}$$

$$\therefore e = \frac{b + a}{2}$$

$$\therefore f = c$$

Nyquist:

Nyquist's theorem states that the minimum sampling frequency required to recreate a waveform is greater than twice the frequency of the highest component of the waveform. If you try to sample a wave at less than its Nyquist frequency then when you reconstruct the wave you will instead get an aliased version of your initial waveform that appears identical to a lower frequency wave.

Exercise:

As an example try to sample and reconstruct the following waves using a 5Hz ($x[N] = \sin(10\pi N)$) sampler

- a) A 2Hz sine wave ($x_1 = \sin 4\pi t$)
- b) A 3Hz sine wave ($x_2 = \sin 6\pi t$)
- c) A 7Hz sine wave ($x_3 = \sin 12\pi t$)
- d) A 8Hz sine wave ($x_4 = \sin 16\pi t$)
- e) A 12Hz sine wave ($x_5 = \sin 22\pi t$)

If you have done it right you should find that when sampled a 5Hz then

$$x_1[N] = x_3[N] = x_5[N] = -x_2[N] = -x_4[N]$$

When using a digital sampler because of aliasing it is important to have a low pass filter with a cut-off filter at the Nyquist frequency to avoid high frequency noise from affecting reading.

It is important to note that $-\sin(x) = \sin(x + 180)$ so that a negative signal can easily be confused for a positive signal with a phase shift.

Circuit Analysis Revision:

RLC can be easily solved by converting all components and inputs to the Laplace domain, using standard circuit analysis techniques to find the output and then converting back to the time domain.

The easiest way to do this is with a Laplace transform table and remembering that the impedance of capacitors and inductors in the Laplace domain are $\frac{1}{Cs}$ and Js respectively.