ELEC 3004/7312: Digital Linear Systems: Signals & Control

Tutorial 5 (Week 10): System Poles Zeros and PID Control

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Review: System Poles and Zeros

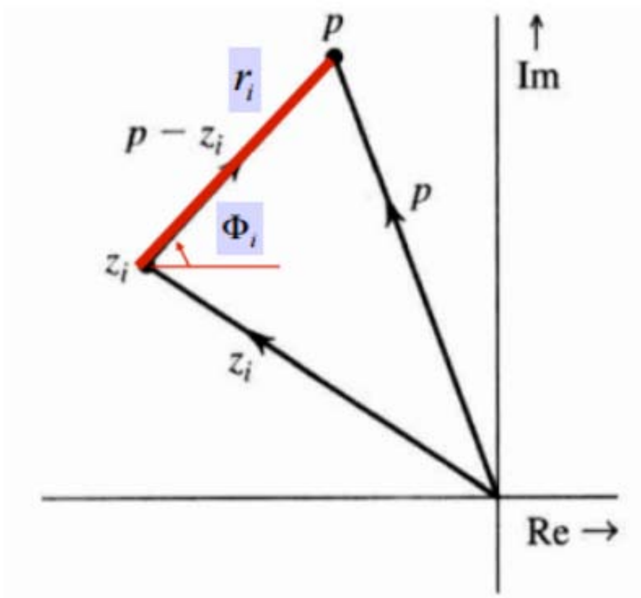
A general system transfer function can be expressed in a poles-zeros form:

$$G(s) = \frac{B(s)}{A(s)} = k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}$$

The value of the transfer function at a complex frequency $s = p$ is:

$$G(s)|_{s=p} = k\frac{(p-z_1)(p-z_2)\cdots(p-z_m)}{(p-\lambda_1)(p-\lambda_2)\cdots(p-\lambda_n)}$$

Take the i th term of the numerator $p - z_i$ for example, it is the vector subtraction between vector p and vector z_i. This vector $p - z_i$ can be illustrated graphically as:



Given the vector length r_i and the angle Φ_i, the expression in polar coordinates is:

$$p - z_i = r_i e^{j\Phi_i}.$$

The transfer function thus can be written as:

$$G(s)|_{s=p} = k\frac{(r_1 e^{j\Phi_1})(r_2 e^{j\Phi_2})\cdots(r_m e^{j\Phi_m})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_n e^{j\theta_n})}$$

Therefore, when the complex frequency s is close to one of the zeros z_i, the amplitude of G is a small value due to the small r_i value; when the complex frequency s is close to one of the poles, the amplitude of G is a large value (tends to infinity) due to a $d_j$value which tends to zero.

Exercise 1: A second-order system has a pair of complex conjugate poles a s = −2±j3 and a single zero at the origin of the s-plane. Find the transfer function and use the pole-zero plot to evaluate the transfer function at s =0+ j5. (The phase response can be ignored in this question)

Solution: From the problem description

$$\begin{aligned} H(s) &= K\frac{s}{(s-(-2+j3))(s-(-2-j3))} \\ &= K\frac{s}{s^2+4s+13} \end{aligned} \qquad (27)$$

The pole-zero plot is shown in Fig. 6. From the figure the transfer function is

$$\begin{aligned} |H(s)| &= K\frac{\sqrt{(0-5)^2}}{\sqrt{(0-(-2))^2+(5-3)^2}\sqrt{(0-(-2))^2+(5-(-3))^2}} \\ &= K\frac{5}{4\sqrt{34}} \end{aligned} \qquad (28)$$

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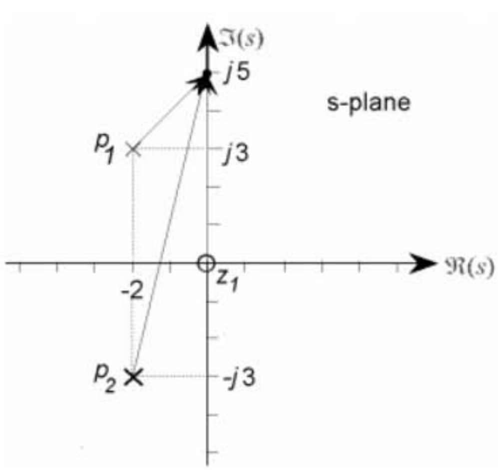
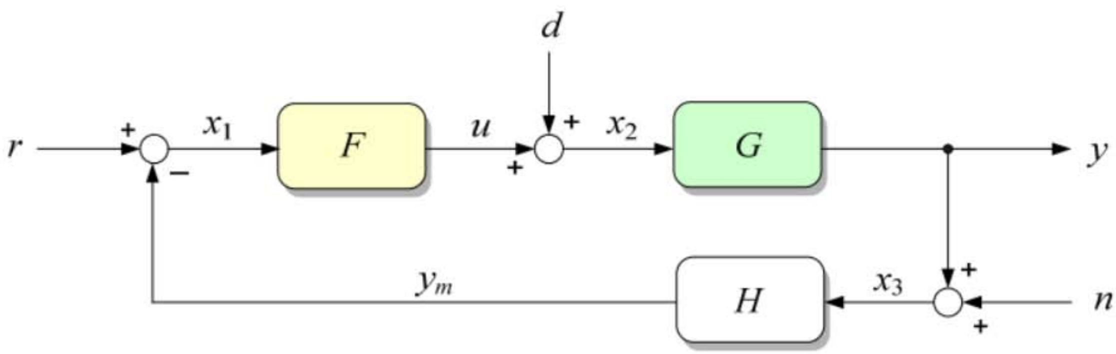


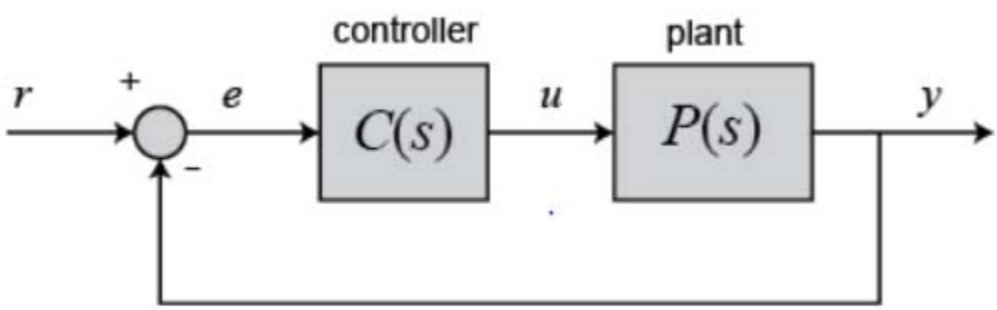
Basic Feedback Equations

Exercise 2: Write the expression of y using r, $d$and n. (Hint: It may be helpful to define intermediate variables, $x_{1,2,3}$ in this case, for the output of the nodes with signal addition/subtraction)a



A short PID control review and a simple exercise

For a unity feedback system:



The output of a PID controller in the time domain is as follows:

$$u(t) = K_p e(t) + K_i \int e(t)dt + K_p \frac{de}{dt}$$

The variable (e) represents the tracking error, the difference between the desired input value (r) and the actual output (y). This error signal (e) will be sent to the PID controller, and the controller computes both the derivative and the integral of this error signal. The control signal (u) to the plant is equal to the proportional gain (K_p) times the magnitude of the error plus the integral gain (K_i) times the integral of the error plus the derivative gain (K_d) times the derivative of the error.

The transfer function of a PID controller is

$$K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}$$

K_p = Proportional gain K_i = Integral gain K_d = Derivative gain.

The Characteristics of P I D Controllers

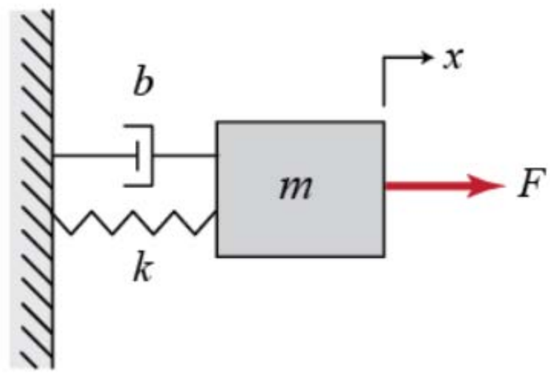
A proportional controller (K_p) will have the effect of reducing the rise time and will reduce but never eliminate the steady-state error. An integral control (K_i) will have the effect of eliminating the steady-state error for a constant or step input, but it may make the transient response slower. A derivative control (K_d) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.
The effects of each of controller parameters, K_p, K_d, and K_i on a closed-loop system are summarized in the table below.

CLOSE LOOP RESPONSE	RISE TIME	OVERSHOOT	SETTLING TIME	S-S ERROR
K_p	Decrease	Increase	Small Change	Decrease
K_i	Decrease	Increase	Increase	Eliminate
K_d	Small Change	Decrease	Decrease	No Change

Note that these correlations may not be exactly accurate, because K_p, K_i, and K_d are dependent on each other. In fact, changing one of these variables can change the effect of the other two. For this reason, the table should only be used as a reference when you are determining the values for K_i, K_p and K_d.

A simple exercise:

Consider a simple mass-spring-damper system with input F and output x (the displacement of the mass):



1. What is the transfer function between X(s) and F(s)?
2. Let m=1kg, b=10Ns/m, k=20N/m, what is the closed-loop transfer function with a proportional controller (Kp), with a proportional-integral (PI) controller (Kp, Ki) and with a proportional-integral-derivative controller (Kp, Ki, Kd)?

Answer to the example above exercise:

1. The modeling equation of this system is

$$M\ddot{x} + b\dot{x} + kx = F$$

Taking the Laplace transform of the modeling equation, we get

$$Ms^2X(s) + bsX(s) + kX(s) = F(s)$$

The transfer function between the displacement $X(s)$ and the input $F(s)$ then becomes

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + bs + k}$$

2. Let
   M = 1 kg
   b = 10 N s/m
   k = 20 N/m
   F = 1 N

Plug these values into the above transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 10s + 20}$$

The closed-loop transfer function of the above system with a proportional controller is:

$$\frac{X(s)}{F(s)} = \frac{K_p}{s^2 + 10s + (20 + K_p)}$$

the closed-loop transfer function with a PI control is:

$$\frac{X(s)}{F(s)} = \frac{K_p s + K_i}{s^3 + 10s^2 + (20 + K_p s + K_i)}$$

The closed-loop transfer function of the given system with a PID controller is:

$$\frac{X(s)}{F(s)} = \frac{K_d s^2 + K_p s + K_i}{s^3 + (10 + K_d)s^2 + (20 + K_p)s + K_i}$$