Signals as Vectors
Systems as Maps

ELEC 3004: Digital Linear Dynamical Systems: Signals & Controls
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Lecture 2
(Makes reference to material from EE263 and ELEC6003)

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Signals as Vectors

- There is a perfect analogy between signals and vectors …

**Signals are vectors!**

- A vector can be represented as a sum of its components in a variety of ways, depending upon the choice of coordinate system. A signal can also be represented as a sum of its components in a variety of ways.
Types of Linear Systems

From Last Week:

- LDS:
  \[
  \dot{x}(t) = A(t)x(t) + B(t)u(t) \\
  y(t) = C(t)x(t) + D(t)u(t)
  \]

- LTI – LDS:
  \[
  \dot{x}(t) = Ax(t) + Bu(t) \\
  y(t) = Cx(t) + Du(t)
  \]
Types of Linear Systems

From Last Week:
- LDS:
  \[
  \dot{x}(t) = A(t)x(t) + B(t)u(t) \\
  y(t) = C(t)x(t) + D(t)u(t)
  \]

To Review:
- Continuous-time linear dynamical system (CT LDS):
  \[
  \frac{dx}{dt} = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)
  \]
- \( t \in \mathbb{R} \) denotes time
- \( x(t) \in \mathbb{R}^n \) is the state (vector)
- \( u(t) \in \mathbb{R}^m \) is the input or control
- \( y(t) \in \mathbb{R}^p \) is the output

Types of Linear Systems

- LDS:
  \[
  \dot{x}(t) = A(t)x(t) + B(t)u(t) \\
  y(t) = C(t)x(t) + D(t)u(t)
  \]

- \( A(t) \in \mathbb{R}^{n \times n} \) is the dynamics matrix
- \( B(t) \in \mathbb{R}^{n \times m} \) is the input matrix
- \( C(t) \in \mathbb{R}^{p \times n} \) is the output or sensor matrix
- \( D(t) \in \mathbb{R}^{p \times m} \) is the feedthrough matrix

⇒ state equations, or “m-input, n-state, p-output’ LDS
Types of Linear Systems

- LDS:

\[
\begin{align*}
\dot{x}(t) &= A(t) x(t) + B(t) u(t) \\
y(t) &= C(t) x(t) + D(t) u(t)
\end{align*}
\]

- \( A(t) \in \mathbb{R}^{n \times n} \) is the dynamics matrix
- \( B(t) \in \mathbb{R}^{n \times m} \) is the input matrix
- \( C(t) \in \mathbb{R}^{p \times n} \) is the output or sensor matrix
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➔ state equations, or “m-input, n-state, p-output’ LDS

Types of Linear Systems

- LDS:

\[
\begin{align*}
\dot{x}(t) &= A(t) x(t) + B(t) u(t) \\
y(t) &= C(t) x(t) + D(t) u(t)
\end{align*}
\]

- **Time-invariant:** where \( A(t), B(t), C(t) \) and \( D(t) \) are **constant**
- **Autonomous:** there is no input \( u \) (\( B,D \) are irrelevant)
- **No Feedthrough:** \( D = 0 \)

- SISO: \( u(t) \) and \( y(t) \) are scalars
- MIMO: \( u(t) \) and \( y(t) \): They’re vectors: Big Deal ?
Discrete-time Linear Dynamical System

- Discrete-time Linear Dynamical System (DT LDS) has the form:

\[ x(t + 1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t) \]

- \( t \in \mathbb{Z} \) denotes time index: \( \mathbb{Z} = \{0, \pm 1, \ldots, \pm n\} \)

- \( x(t), u(t), y(t) \in \) are sequences

- Differentiation handled as difference equation:
  ➞ first-order vector recursion

Signals as Vectors

- Represent them as Column Vectors

\[
x = \begin{bmatrix}
x[1] \\
x[2] \\
x[3] \\
\vdots \\
x[N]
\end{bmatrix}.
\]
Signals as Vectors

- Can represent phenomena of interest in terms of signals

- Natural vector space structure (addition/subtraction/norms)

- Can use norms to describe and quantify properties of signals
Various Types

- Audio signal (sound pressure on microphone)
- B/W video signal (light intensity on photosensor)
- Voltage/current in a circuit (measure with multimeter)
- Car speed (from tachometer)
- Robot arm position (from rotary encoder)
- Daily prices of books / air tickets / stocks
- Hourly glucose level in blood (from glucose monitor)
- Heart rate (from heart rate sensor)

Vector Refresher

- Length: $|x|^2 = x \cdot x$
- Decomposition: $x = c_1 y + e_1 = c_2 y + e_2$
- Dot Product of $\perp$ is 0: $x \cdot y = 0$
Vectors [2]

• Magnitude and Direction

\[ f \cdot x = |f||x| \cos(\theta) \]

• Component (projection) of a vector along another vector

![Diagram of vector components](image)

\[ f = cx + e \quad \text{Error Vector} \]

Vectors [3]

• \(\infty\) bases given \(\vec{x}\)

![Diagram of vector components](image)

• Which is the best one?

\[ e = c(x) \quad (a) \]

\[ c_{x} = \frac{f \cdot x}{|x|} \cos \theta \]

\[ c_{x} = \frac{f \cdot x}{|x|} \cos \theta = f \cdot x \]

\[ e = \frac{f \cdot x}{x \cdot x} \]

\[ f \cdot x = 0 \]

• Can I allow more basis vectors than I have dimensions?
Signals Are Vectors

- A Vector / Signal can represent a sum of its components

  **Remember** (Lecture 5, Slide 10):
  
  \[
  \text{Total response} = \text{Zero-input response} + \text{Zero-state response}
  \]

  \[
  \begin{array}{c|c}
  \text{Initial conditions} & \text{External Input} \\
  \end{array}
  \]

- Vectors are Linear
  - They have **additivity** and **homogeneity**

Vectors / Signals Can Be Multidimensional

- A signal is a quantity that varies as a function of an index set

- They can be multidimensional:
  - 1-dim, discrete index (time): \( x[n] \)
  - 1-dim, continuous index (time): \( x(t) \)
  - 2-dim, discrete (e.g., a B/W or RGB image): \( x[j; k] \)
  - 3-dim, video signal (e.g, video): \( x[j; k; n] \)
It's Just a Linear Map

\[ y[n] = 2u[n] \] is a linear map

BUT \[ y[n] = 2(u[n] - 1) \] is NOT Why?

Because of homogeneity!

\[ T(au) = aT(u) \]

Norms of signals

Can introduce a notion of signals being "nearby."

This is characterized by a metric (or distance function).

\[ d(x, y) \]

If compatible with the vector space structure, we have a norm.

\[ ||x - y|| \]
Examples of Norms

Can use many different norms, depending on what we want to do. The following are particularly important:

- $\ell_2$ (Euclidean) norm:
  \[ \|x\|_2 = \left( \sum_{k=1}^{n} |x[k]|^2 \right)^{\frac{1}{2}} \quad \text{norm}(x,2) \]

- $\ell_1$ norm:
  \[ \|x\|_1 = \sum_{k=1}^{n} |x[k]| \quad \text{norm}(x,1) \]

- $\ell_\infty$ norm:
  \[ \|x\|_\infty = \max_{k} |x[k]| \quad \text{norm}(x,\infty) \]

What are the differences?

Properties of norms

For any norm $\| \cdot \|$, and any signal $x$, we have:

- Linearity: if $C$ is a scalar,
  \[ \|C \cdot x\| = |C| \cdot \|x\| \]

- Subadditivity (triangle inequality):
  \[ \|x + y\| \leq \|x\| + \|y\| \]

Can use norms:

- To detect whether a signal is (approximately) zero.
- To compare two signals, and determine if they are “close.”
  \[ \|x - y\| \approx 0 \]
A signal may be thought of as having components.

Component of a Signal

\[ f(t) \approx c x(t) \quad t_1 \leq t \leq t_2 \]
\[ c = \frac{\int_{t_1}^{t_2} f(t)x(t) \, dt}{\int_{t_1}^{t_2} x^2(t) \, dt} = \frac{1}{E_x} \int_{t_1}^{t_2} f(t)x(t) \, dt \]

- Let’s take an example:

\[ f(t) = c \sin t \quad 0 \leq t \leq 2\pi \]
\[ x(t) = \sin t \quad \text{and} \quad E_x = \int_0^{2\pi} \sin^2(t) \, dt = \pi \]

Fig. 3.3 Approximation of square signal in terms of a single sinusoid.

Thus

\[ f(t) \approx \frac{4}{\pi} \sin t \quad (3.14) \]
Basis Spaces of a Signal

\[
\int_{t_1}^{t_2} x_m(t)x_n(t)\,dt = \begin{cases} 
0 & m \neq n \\
E_n & m = n
\end{cases}
\]

\[f(t) \approx c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t)\]

\[= \sum_{n=1}^{N} c_n x_n(t)\]

\[e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t)\]

\[c_n = \frac{\int_{t_1}^{t_2} f(t)x_n(t)\,dt}{\int_{t_1}^{t_2} x_n^2(t)\,dt}\]

\[= \frac{1}{E_n} \int_{t_1}^{t_2} f(t)x_n(t)\,dt \quad n = 1, 2, \ldots, N\]

\[f(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + \cdots\]

\[= \sum_{n=1}^{\infty} c_n x_n(t) \quad t_1 \leq t \leq t_2\]

- Observe that the error energy \(Ee\) generally decreases as \(N\), the number of terms, is increased because the term \(Ck \, E_k\) is nonnegative. Hence, it is possible that the error energy \(\rightarrow 0\) as \(N \rightarrow \infty\). When this happens, the orthogonal signal set is said to be complete.
- In this case, it’s no more an approximation but an equality
Linear combinations of signals

Application Example: Active Noise Cancellation

A “noise” signal, that we want to get rid of.

- At subject location, signal is
  \[ x[n] \]

- Microphone picks up signal
  \[ x_c[n] \]

- Subtract the two signals:
  \[ y(t) = x(t) - x_c(t) \]

Notice careful synchronization is needed!
Where are we going with this?

Consider the following system:

- How to model and predict (and control the output)?

Source: EE263 (s.1-13)
Consider the following system:

- $x(t) \in \mathbb{R}^8$, $y(t) \in \mathbb{R}^1 \rightarrow 8$-state, single-output system
- Autonomous: No input yet! ($u(t) = 0$)
This can help simplify matters…
An Example

- Consider the following system:
Example: Let’s consider the control...

Expand the system to have a control input...

- $B \in \mathbb{R}^{8 \times 2}$, $C \in \mathbb{R}^{2 \times 8}$ (note: the 2nd dimension of $C$)

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = 0$$

- Problem: Find $u$ such that $y_{des}(t) = (1, -2)$

- A simple (and rational) approach:
  - solve the above equation!
  - Assume: static conditions ($u, x, y$ constant)

$\Rightarrow$ Solve for $u$:

$$u_{static} = (-CA^{-1}B)^{-1}y_{des} = \begin{bmatrix} -0.63 \\ 0.36 \end{bmatrix}$$

Example: Apply $u = u_{static}$ and presto!

- Note: It takes 1500 seconds for the $y(t)$ to converge ...
  but that’s natural ... can we do better?

Source: EE263 (s.1-13)
Example: Yes we can!

• How about:

\[ y_2(t) \]

\[ y_1(t) \]

Example: How? How about a more clever input?

• How about:

\[ y_1(t) \]

\[ y_2(t) \]

• Converges in 50 seconds (3.3% of the time 😊)

Source: EE263 (s.1-13)
Example: Can we beat it? Larger inputs & LDS

- Converges in 20 seconds (1.3% of the time 😊)

Dynamical Systems...

- A system with a memory
  - Where past history (or derivative states) are relevant in determining the response
- Ex:
  - RC circuit: Dynamical
    - Clearly a function of the “capacitor’s past” (initial state) and
    - Time! (charge / discharge)
  - R circuit: is memoryless – the output of the system
    (recall V=IR) at some time \( t \) only depends on the input at time \( t \)

- Lumped/Distributed
  - Lumped: Parameter is constant through the process
    & can be treated as a “point” in space
  - Distributed: System dimensions ≠ small over signal
    - Ex: waveguides, antennas, microwave tubes, etc.
Causality:
Causal (physical or nonanticipative) systems

- Is one for which the output at any instant \( t_0 \) depends only on the value of the input \( x(t) \) for \( t \leq t_0 \). Ex:
  \[
  u(t) = x(t-2) \Rightarrow \text{causal} \quad \text{or} \quad u(t) = x(t-2) + x(t+2) \Rightarrow \text{noncausal}
  \]

- A "real-time" system must be causals
  - How can it respond to future inputs?
- A prophetic system: knows future inputs and acts on it (now)
  - The output would begin before \( t_0 \)
- In some cases Noncausal maybe modelled as causal with delay
- Noncausal systems provide an upper bound on the performance of causal systems

\[ \text{ELEC 3004: Systems} \quad 9 \text{ March 2015} \quad \text{43} \]
Then a System is a **MATRIX**

\[ y = Du. \]

\[
\begin{bmatrix}
y[1] \\
y[2] \\
\vdots \\
y[M]
\end{bmatrix}
= 
\begin{bmatrix}
D_{11} & D_{12} & \cdots & D_{1N} \\
D_{21} & D_{22} & \cdots & D_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
D_{M1} & D_{M2} & \cdots & D_{MN}
\end{bmatrix}
\begin{bmatrix}
u[1] \\
u[2] \\
\vdots \\
u[N]
\end{bmatrix}.
\]

\[ y[i] = \sum_j D_{ij}u[j]. \]
Linear Time Invariant

- Linear & Time-invariant (of course - tautology!)
- Impulse response: \( h(t) = F(\delta(t)) \)
- Why?
  - Since it is linear the output response (\( y \)) to any input (\( x \)) is:
  
  \[
  \begin{align*}
  x(t) &= \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau \\
  y(t) &= F \left[ \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau \right] = \int_{-\infty}^{\infty} x(\tau) F(\delta(t-\tau)) \, d\tau \\
  h(t) &= F(\delta(t)) \\
  \Rightarrow y(t) &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau = x(t) * h(t)
  \end{align*}
  \]

- The output of any continuous-time LTI system is the convolution of input \( u(t) \) with the impulse response \( F(\delta(t)) \) of the system.

Linear Dynamic [Differential] System

\( \equiv \) LTI systems for which the input & output are linear ODEs

\[
\begin{align*}
a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} &= b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}
\end{align*}
\]

Laplace:

\[
\begin{align*}
A(s) Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) &= b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s) \\
\end{align*}
\]

- Total response = Zero-input response + Zero-state response

  \[
  \text{Initial conditions} \quad \text{External Input}
  \]
Linear Systems and ODE’s

• Linear system described by differential equation

\[ a_0y + a_1\frac{dy}{dt} + \cdots + a_n\frac{d^n y}{dt^n} = b_0x + b_1\frac{dx}{dt} + \cdots + b_m\frac{d^m x}{dt^m} \]

• Which using Laplace Transforms can be written as

\[ a_0Y(s) + a_1sY(s) + \cdots + a_ns^nY(s) = b_0X(s) + b_1sX(s) + \cdots + b_ms^mX(s) \]

\[ A(s)Y(s) = B(s)X(s) \]

where \( A(s) \) and \( B(s) \) are polynomials in \( s \)

Unit Impulse Response

• \( \delta(t) \): Impulsive excitation
• \( h(t) \): characteristic mode terms

**Ex:**

Determine the unit impulse response \( h(t) \) for a system specified by the equation

\[ (s^2 + 3s + 2) y(t) = x(t) \] \( (2.23) \)

This is a second-order system \((n = 2)\) having the characteristic polynomial

\[ s^2 + 3s + 2 = (s + 1)(s + 2) \]

The characteristic roots of this system are \( s = -1 \) and \( s = -2 \). Therefore

\[ Y(s) = \frac{X(s)}{(s + 1)(s + 2)} \] \( (2.26a) \)

Differentiation of this equation yields

\[ Y^{(n)}(s) = \frac{X^{(n)}(s)}{(s + 1)(s + 2)} \] \( (2.26b) \)

The initial conditions are basic \( P_{0j} \) \((j = 1, 2)\) for \( n = 2 \)

\[ \begin{align*}
| & \text{at} \ t = 0 \ \text{and} \ t > 0 \ \text{for} \ s = -1 \\
& \text{at} \ t = 0 \ \text{and} \ t > 0 \ \text{for} \ s = -2
\end{align*} \]

Based on Eqs. (2.26a) and (2.26b), and substituting the initial conditions just given, we obtain

\[ \delta \left( t \right) = \delta \left( 0^+ \right) - 2 \delta \left( 0^- \right) \]

\( 1 \ \text{is basic} \ \text{for} \ s = -1 \)

\( 1 \ \text{is basic} \ \text{for} \ s = -2 \)

\[ \begin{align*}
& \text{Solution of these two simultaneous equations yields} \\
& \text{At} \ 0 \ \text{for} \ s = -1 \ \text{and} \ t > 0
\end{align*} \]

\[ \begin{align*}
& \delta(t) = e^{-t} - 2e^{-2t} \\
& \text{At} \ t = 0 \ \text{for} \ s = -2 \ \text{and} \ t > 0
\end{align*} \]

\[ \begin{align*}
& \text{Therefore, according to Eq. (2.26b), \ \text{L.C.} + 2 \ \text{so that} \\
& \text{L.C.} \left[ \delta(t) \right] = \delta(t) = \delta \left( 0^+ \right) - 2 \delta \left( 0^- \right)
\end{align*} \]

Also in this case, \( \delta(t) \) is the second-order term absent in PDE. Therefore

\[ h(t) = \delta(t) \]
System Models

- Various things – all the same!

### Table 2.1 Summary of Through- and Across-Variables for Physical Systems

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### Circuits

$$\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{1}{C_1C_2R_1R_2s^2 + C_2(R_1 + R_2)s + 1}$$
Motors

5. DC motor, field-controlled, rotational actuator

\[ \theta(s) \quad V_f(s) = \frac{K_m}{s(Js + b)(L_f s + R_f)} \]

7. AC motor, two-phase control field, rotational actuator

\[ \theta(s) \quad V_f(s) = \frac{K_m}{s(\tau s + 1)} \]
\[ \tau = J/(b - m) \]
\[ m = \text{slope of linearized torque-speed curve (normally negative)} \]

Mechanical Systems

15. Accelerometer, acceleration sensor

\[ x_a(t) = y(t) - x_a(t), \]
\[ X_a(s) = \frac{\omega^2}{s^2 + (b/M)s + k/M} \]

For low-frequency oscillations, where \( \omega < \omega_n \),
\[ X_a(j\omega) \approx \frac{\omega^2}{k/M} \]
Thermal Systems

16. Thermal heating system

\[ q(s) = C_s \tau + (Qs + 1/R_i) \]

where

\[ \tau = \tau_s - \tau_i = \text{temperature difference due to thermal process} \]

\[ C_s = \text{thermal capacitance} \]

\[ Q = \text{fluid flow rate = constant} \]

\[ S = \text{specific heat of water} \]

\[ R_i = \text{thermal resistance of insulation} \]

\[ q(s) = \text{transform of rate of heat flow of heating element} \]

First Order Systems

**First order systems**

\[ ay' + by = 0 \quad (a \neq 0) \]

Righthand side is zero:
- called *autonomous system*
- solution is called *natural or unforced response*

Can be expressed as

\[ Ty' + y = 0 \quad \text{or} \quad y' + ry = 0 \]

where
- \( T = a/b \) is a *time* (units: seconds)
- \( r = b/a = 1/T \) is a *rate* (units: 1/sec)
First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T(sY(s) - y(0)) + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT + 1} = \frac{y(0)}{s + 1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, $y$ decays exponentially

- $T$ gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half ($0.693 = \log 2$)
- $4.6T$ gives time to decay by 0.01 ($4.6 = \log 100$)

if $T < 0$, $y$ grows exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100
First Order Systems

Examples

simple RC circuit:

\[ \begin{array}{c}
R \\
\hline \\
C \\
\hline \\
v
\end{array} \]

+ circuit equation: \( RCv' + v = 0 \)

solution: \( v(t) = v(0)e^{-t/(RC)} \)

population dynamics:

- \( y(t) \) is population of some bacteria at time \( t \)
- growth (or decay if negative) rate is \( y' = by - dy \) where \( b \) is birth rate, \( d \) is death rate
- \( y(t) = y(0)e^{(b-d)t} \) (grows if \( b > d \); decays if \( b < d \))

Second Order Systems

Second order systems

\[ ay'' + by' + cy = 0 \]

assume \( a > 0 \) (otherwise multiply equation by \(-1\))

solution by Laplace transform:

\[ a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 0 \]

solve for \( Y \) (just algebraic)

\[ Y(s) = \frac{asy(0) + ag(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c} \]

where \( \alpha = ag(0) \) and \( \beta = ag'(0) + by(0) \)
Second Order Systems

so solution of \( ay'' + by' + cy = 0 \) is

\[
y(t) = \mathcal{L}^{-1}\left( \frac{\alpha s + \beta}{\alpha s^2 + bs + c} \right)
\]

- \( \chi(s) = \alpha s^2 + bs + c \) is called characteristic polynomial of the system
- form of \( y = \mathcal{L}^{-1}(Y) \) depends on roots of characteristic polynomial \( \chi \)
- coefficients of numerator \( \alpha s + \beta \) come from initial conditions

Example: Speaking of Circuits

\[
C \frac{dV_o(t)}{dt} = \frac{V_i(t) - V_o(t)}{R}.
\]

Source: ELEC 3004 (p.3-42)
What about the DIGITAL case?

- Is it still linear?

Source: [SIROcco (p. 3-46)]

What about the DIGITAL case?

- Can LDS help do better than quantization?
What about the DIGITAL case?

• Problem:
  Estimate signal $u$, given quantized, filtered signal $y$

• Some solutions:
  – ignore quantization
  – design equalizer $G(s)$ for $H(s)$ (i.e., $GH \cong 1$)
  – approximate $u$ as $G(s)y$

  ➞ Pose as an estimation problem

Source: EE263 (s.1-124)

What about the DIGITAL case?

• RMS error 0.03, well below quantization error (!)

Source: EE263 (s.1-124)
Ex: Deblurring

- Matlab: `deconvwnr`

What about …
• For small current inputs, neuron membrane potential output response is surprisingly **linear**.

• Though this has limits … neurons “spike” are (quite) nonlinear (truly)

Source: **ELEC 3004** (p. 3-49)

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**Next Time…**

• **Sampling**
  – Measurements at regular intervals of a continuous signal
  – Not to be confused with
    “How to try regional dishes without indigestion”

• Review:
  – Chapter 8 of Lathi

• Send (and you shall receive) a positive signal 😊