An Introduction to Signals & Systems

ELEC 3004: Digital Linear Systems Signals & Controls
Dr. Surya Singh

Week 1 – Thursday Open Lecture 1 [WARNING: EXPERIMENTAL!]

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<td>24</td>
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</tr>
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</table>
Signals

What is a Signal?

- Communicates information
- Varies in Time and/or Space

Everywhere:
- The type
- The screen
- Your 5 senses
- Telephony
- Interest rates
- Leaf colour

\[ F(x) \to \text{Signal} \]

Temperature
Distance
Force
Speed
## Understanding & Classifying Signals

<table>
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<th>Metrics</th>
<th>Classifications</th>
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<td>Size</td>
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<tr>
<td>Signal Power</td>
<td>Analog</td>
</tr>
<tr>
<td>Frequency</td>
<td>Digital</td>
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<tr>
<td>Phase</td>
<td>Even</td>
</tr>
<tr>
<td>Entropy</td>
<td>Odd</td>
</tr>
<tr>
<td>- Deterministic signals</td>
<td></td>
</tr>
<tr>
<td>- Random signals</td>
<td></td>
</tr>
</tbody>
</table>

### Signal Size

- The size of any entity is a number that indicates the largeness or strength of that entity. Generally speaking, the signal **amplitude** varies with **time**.

\[
S = \int_0^T u(t) \, dt
\]

- However, this will be a defective measure because even for a large signal \(x(t)\), its positive and negative areas could cancel each other, indicating a signal of small size.
Signal Energy

- Consider the area under a signal $x(t)$ as a possible measure of its size, because it takes account not only of the amplitude but also of the duration.

- Instead we look at its energy or signal energy

$$E = \int_{-\infty}^{\infty} u^2(t) \, dt$$

- But this can be infinite!
- Not to be confused with mechanical Energy -- Not in Joules

Signal Power:

- Generalize this to a finite measure via a RMS (root means square measurand)

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |u(t)|^2 \, dt$$

- Not to be confused with mechanical Power -- Not in Watts
Signal Classifications

- Classifications
  - A “pair-wise” way of characterizing the signal by putting it into a “descriptive bin”
  - Ex: How to describe a person – well we could measure them (weight, height, power, etc.) or we could sort by category (“North/South-side,” Australian, etc.) ➔ Neither is perfect

- Some common classifications:
  1. Continuous-time and discrete-time signals
  2. Analog and digital signals
  3. Periodic and aperiodic signals
  4. Real and complex signals
  5. Deterministic and probabilistic signals

Continuous-Time

- The independent variable (x-axis) – in this case t – is continuous (which may be the ℝeals, but can be others)
- This does not dictate the form of u(t) -- it may be either continuous or discrete.
The independent variable (x-axis) – in this case \( t \) – is continuous (which may be the \( \mathbb{R} \) reals, but can be others).

This does not dictate the form of \( u(t) \) -- it may be either continuous or discrete.

Continuous Time

- The independent variable is only set for fixed, discrete values
- Ex: \( t \in \) Integers
- Written in [Square Brackets]
- \( u[t_n] = u[n] \)
- \( t_n = \{t_0, t_1, t_2, \ldots, t_n\} \)
- Relationship to continuous time: \( u[n] = u(nT_s) \)

Discrete Time

- \( T \) = Sample Period
- \( T = \) Time between samples
- \( t \notin \mathbb{Z} \) (Integers)
• Continuous and Discrete on the $y$-axis
• $u(t) = \text{Continuous } \Rightarrow \text{Analog}$
• $u(t) = \text{Discrete } \Rightarrow \text{Digital}$
  – Number of levels can be many
  – Simplest case is for two $\Rightarrow$ Binary digit (or bit) signal

• (a) analog, continuous time | (b) digital, continuous time
• (c) analog, discrete time and (d) digital, discrete time
Periodic and Aperiodic Signals

- A signal \( x(t) \) is *periodic* if for some positive constant \( T_0 \)
  \[
  x(t) = x(t + T_0)
  \]

- For periodic signals \( x(t) \) of period \( T_0 \):
  - the area under \( x(t) \) over any interval of duration \( T_0 \) is the same
  \[
  A = \int_{a}^{a+T_0} u(t) \, dt = \int_{b}^{b+T_0} u(t) \, dt
  \]

Real and Complex Signals

\[
\begin{align*}
  x(t) &= x_1(t) + x_2(t) \, j \\
  s &= x_1 + x_2 \, j \\
  \rightarrow e^{st} &= e^{(x_1 + x_2 \, j)} = e^{x_1 \, t} e^{x_2 \, t \, j} \\
  &= e^{x_1 \, t} (\cos(x_2 \, t) + \sin(x_2 \, t) \, j)
\end{align*}
\]
Deterministic and Random

- Deterministic
  - are those whose description is known completely

- Random
  - Those signals whose descriptions is unknown incompletely via statistical or probabilistic descriptions.

Ergodicity

1. **Over time**: multiple readings of a quantity over time
   - “stationary” or “ergodic” system
   - Sometimes called “integrating”

2. **Over space**: single measurement (summed) from multiple sensors each distributed in space

3. **Same Measurand**: multiple measurements take of the same observable quantity by multiple, related instruments
   - e.g., measure position & velocity simultaneously
   - Basic “sensor fusion”
Classifying Signals: Even and Odd

- **Even**
  \[ u_{\text{even}}(t) = u_{\text{even}}(-t) \]

  \[ \int_{-a}^{a} u_e(t) \, dt = 2 \int_{0}^{a} u_e(t) \, dt \]

  Every signal can be expressed as a sum of even and odd components

- **Odd**
  \[ u_{\text{odd}}(t) = -u_{\text{odd}}(-t) \]

  \[ \int_{-a}^{a} u_e(t) \, dt = 0 \]

Signal Models

- A “model” of signals
- Key functions include:
  - Step
  - Impulse
  - Exponential functions play

- Basis for representing other signals,
- Simplify many aspects of the signals and systems.
Signal Models

1] Unit Step

- $u(t) = \begin{cases} A & t > t_0 \\ 0 & t < t_0 \end{cases}$
- if $A=1$, $t_0=0$
  - Heaviside function

2] Unit Impulse

- $\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$
- Can be approximated as:
  $$\delta_a(x) = \lim_{a \to 0} \frac{1}{a \sqrt{\pi}} e^{-x^2/a^2}$$

Signal Models: Multiplication by Impulse

- Multiply $F(t)$ by $\delta(t)$
- Is about $t=0$:
  - $F(t)$ is continuous at $t=0$
  - $\delta$ only has value at $t=0$
  - $\mathbb{Z}=F(0)=F(t=0)$
- Sampling Property of the Impulse:
  $$F(t) \delta(t) = F(0) \delta(t) \quad \rightarrow \quad F(t) \delta(t-T) = F(T) \delta(t-T)$$
  $$\int_{-\infty}^{\infty} F(t) \delta(t) \, dt = F(0) \int_{-\infty}^{\infty} \delta(t) \, dt = \mathbb{Z}$$
  $$\int_{-\infty}^{\infty} F(t) \delta(t-T) \, dt = F(T) \int_{-\infty}^{\infty} \delta(t) \, dt = F(T)$$
Signal Models

3] Sinusoidal & Exponential Signals

\[ u(t) = A \cos(\omega t + \phi) \]

- \( A, \omega \): Amplitude and Frequency (\( \omega = 2\pi f \))
- \( \phi \): phase angle

Figure 1.21: Sinusoids of complex frequency \( \sigma + j\omega \).
Further Classifications of Systems

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Modelling order

- Modelling order depends on what you are trying to achieve
Modelling Ties Back with ELEC 2004

- Linear Circuit Theorems, Operational Amplifiers
- Operational Amplifiers
- Capacitors and Inductors, RL and RC Circuits
- AC Steady State Analysis
- AC Power, Frequency Response
- Laplace Transform
- Reduction of Multiple Sub-Systems
- Fourier Series and Transform
- Filter Circuits

→ Modelling Tools!

Example: RC Circuits

\[ y(t) = Rf(t) + \frac{1}{C} \int_{-\infty}^{t} f(\tau) \, d\tau \]
\[ y(t) = Rf(t) + \frac{1}{C} \int_{-\infty}^{0} f(\tau) \, d\tau + \frac{1}{C} \int_{0}^{t} f(\tau) \, d\tau \]
\[ y(t) = v_C(0) + Rf(t) + \frac{1}{C} \int_{0}^{t} f(\tau) \, d\tau \]
\[ y(t) = v_C(t_0) + Rf(t) + \frac{1}{C} \int_{t_0}^{t} f(\tau) \, d\tau \]
First Order RC Filter

- Passive, First-Order Resistor-Capacitor Design:

\[ T(s) = \frac{a_1 s + a_0}{s + \omega_0} \]

- 3dB (½ Signal Power):

\[ \omega = 2\pi f \]
\[ f_c = \frac{1}{2\pi RC} \]

- Magnitude:

\[ |V_{out}| = \sqrt{\frac{1}{\omega RC}} |V_{in}| \]

- Phase:

\[ \phi = \tan^{-1} (-\omega RC') \]

Example of 2nd Order: RLC Circuits

- KCL:

\[ \frac{V_s(t) - V_C(t)}{R_3} = C \frac{d}{dt} V_C(t) + i(t) \]

- KVL:

\[ V_C(t) = L \frac{d}{dt} i(t) + R_2 i(t) \]

- Combining:

\[ V_s(t) = R_1 LC \frac{d^2}{dt^2} i(t) + (L + R_1 R_2 C) \frac{d}{dt} i(t) + (R_1 + R_2) i(t) \]
2nd Order Active RC Filter (Sallen–Key)

- 2nd Order System Sallen–Key Low-Pass Topology:

![Sallen–Key Filter Diagram]

\[ \frac{v_{\text{in}} - v_x}{R_1} = C_1 s (v_x - v_{\text{out}}) + \frac{v_x - v_{\text{out}}}{R_2} \]

- **KCL:**

\[ \frac{v_{\text{in}} - v_x}{R_1} = C_1 s (v_x - v_{\text{out}}) + \frac{v_x - v_{\text{out}}}{R_2} \]

- Combined with Op-Amp Law:

\[ \frac{v_{\text{in}} - v_{\text{out}}}{R_1} = C_1 s v_{\text{out}} (C_2 R_2 + 1) - v_{\text{out}} + \frac{v_{\text{out}} (C_2 R_2 + 1) - v_{\text{out}}}{R_2} \]

- Solving for gives a 2nd order System:

\[ \frac{v_{\text{out}}}{v_{\text{in}}} = \frac{1}{C_1 C_2 R_1 R_2 s^2 + C_2 (R_1 + R_2) s + 1} \]

Another 2nd Order System:
Accelerometer or Mass Spring Damper (MSD)

- General accelerometer:
  - Linear spring (k) (0th order w/r t o)
  - Viscous damper (b) (1st order)
  - Proof mass (m) (2nd order)

- Electrical system analogy:
  - resistor (R) : damper (b)
  - inductance (L) : spring (k)
  - capacitance (C) : mass (m)

![Accelerometer Diagram]

Build this for Real in ELEC 4403

ELEC 3004: Systems 6 March 2014: 34
Measuring Acceleration:
Sense \( a \) by measuring spring motion \( Z \)

- Start with Newton’s 2\textsuperscript{nd} Law:
  \[ ma = F \]

- Substitute:
  \[ m \frac{d^2 x}{dt^2} = k (X - x) + b \frac{d (X - x)}{dt} \]

\( Z \equiv (X - x) \rightarrow x = X - Z \)

\[ m \frac{d^2 x}{dt^2} = m \frac{d^2 Z}{dt^2} + kZ + b \frac{dZ}{dt} \]

- Solve ODE:
  \[ X(t) = X_0 e^{i\omega t} \quad Z(t) = Z_0 e^{i\omega t} \]

Measuring Acceleration [2]

- Substitute candidate solutions:
  \[ m \frac{d^2 (X_0 e^{i\omega t})}{dt^2} = m \frac{d^2 (Z_0 e^{i\omega t})}{dt^2} + k \left( Z_0 e^{i\omega t} \right) + b \frac{d(Z_0 e^{i\omega t})}{dt} \]

\[ -m\omega^2 X_0 e^{i\omega t} = -m\omega^2 Z_0 e^{i\omega t} + kZ_0 e^{i\omega t} + (i\omega) bZ_0 e^{i\omega t} \]

- Define Natural Frequency \((\omega_0)\)
  & Simplify for \( Z_0 \)
  (the spring displacement “magnitude”):

\[ \omega_0 \equiv \sqrt{\frac{k}{m}} \]

\[ Z_0 = \frac{m\omega^2 X_0}{m\omega^2 - k - i\omega b} = \frac{X_0}{\sqrt{1 - \omega_0^2 - \frac{b^2}{m^2\omega^2}}} \]
Acceleration: 2nd Order System

- Plot for a “unit” mass, etc.

- For $\omega < \omega_0$:
  \[ Z_0 \approx \frac{\omega^2 x_0}{\omega_0^2} = \frac{a}{\omega_0^2} \]
  \[ \rightarrow a = Z_0 \omega_0^2 \]
  ➔ it’s an Accelerometer

- For $\omega \sim \omega_0$
  - As: $b \rightarrow 0$, $Z \rightarrow \infty$
  - Sensitivity $\uparrow$

- For $\omega > \omega_0$:
  \[ Z_0 \approx X_0 \]
  ➔ it’s a Seismometer

Calculates derivatives of $L$ to find terms... equations... I DON’T KNOW HOW!??!

I’M AN ELECTRICAL ENGINEER!!
# Equivalence Across Domains

## Table 2.1 Summary of Through- and Across-Variables for Physical Systems

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<tr>
<th>System</th>
<th>Variable Through Element</th>
<th>Integrated Through Variable</th>
<th>Variable Across Element</th>
<th>Integrated Across Variable</th>
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<tr>
<td>Electrical</td>
<td>Current, $i$</td>
<td>Charge, $q$</td>
<td>Voltage difference, $v_{21}$</td>
<td>Flux linkage, $\lambda_{21}$</td>
</tr>
<tr>
<td>Mechanical translational</td>
<td>Force, $F$</td>
<td>Translational momentum, $P$</td>
<td>Velocity difference, $v_{21}$</td>
<td>Displacement difference, $y_{21}$</td>
</tr>
<tr>
<td>Mechanical rotational</td>
<td>Torque, $T$</td>
<td>Angular momentum, $h$</td>
<td>Angular velocity difference, $\omega_{21}$</td>
<td>Angular displacement difference, $\theta_{21}$</td>
</tr>
<tr>
<td>Fluid</td>
<td>Fluid volumetric rate of flow, $Q$</td>
<td>Volume, $V$</td>
<td>Pressure difference, $P_{21}$</td>
<td>Pressure momentum, $\gamma_{21}$</td>
</tr>
<tr>
<td>Thermal</td>
<td>Heat flow rate, $q$</td>
<td>Heat energy, $H$</td>
<td>Temperature difference, $S_{21}$</td>
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Source: Dorf & Bishop, Modern Control Systems, 12th Ed., p. 73

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## Table 2.2 Summary of Governing Differential Equations for Ideal Elements

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<thead>
<tr>
<th>Type of Element</th>
<th>Physical Element</th>
<th>Governing Equation</th>
<th>Energy $E$ or Power $P$</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inductive storage</td>
<td>Electrical inductance</td>
<td>$v_2 = L \frac{dE}{dt}$</td>
<td>$E = \frac{1}{2} L i^2$</td>
<td>$v_2 \xrightarrow{i} \frac{1}{2} L i^2$</td>
</tr>
<tr>
<td></td>
<td>Translational spring</td>
<td>$v_2 = \frac{1}{k} \frac{dP}{dt}$</td>
<td>$P = \frac{1}{2} k v_2^2$</td>
<td>$v_2 \xrightarrow{P} \frac{1}{2} k v_2^2$</td>
</tr>
<tr>
<td></td>
<td>Rotational spring</td>
<td>$\omega_2 = \frac{1}{k} \frac{dQ}{dt}$</td>
<td>$Q = \frac{1}{2} k \omega_2^2$</td>
<td>$\omega_2 \xrightarrow{Q} \frac{1}{2} k \omega_2^2$</td>
</tr>
<tr>
<td></td>
<td>Fluid inert</td>
<td>$P_2 = \frac{1}{k} \frac{dQ}{dt}$</td>
<td>$Q = \frac{1}{2} k P_2^2$</td>
<td>$P_2 \xrightarrow{Q} \frac{1}{2} k P_2^2$</td>
</tr>
<tr>
<td>Capacitive storage</td>
<td>Electrical capacitance</td>
<td>$i_2 = \frac{1}{C} \frac{dQ}{dt}$</td>
<td>$Q = \frac{1}{2} C i_2^2$</td>
<td>$i_2 \xrightarrow{Q} \frac{1}{2} C i_2^2$</td>
</tr>
<tr>
<td></td>
<td>Translational mass</td>
<td>$F = M \frac{d\Delta x}{dt}$</td>
<td>$\Delta x = \frac{1}{2} F^2$</td>
<td>$F \xrightarrow{\Delta x} \frac{1}{2} F^2$</td>
</tr>
<tr>
<td></td>
<td>Rotational mass</td>
<td>$T = J \frac{d\Delta \theta}{dt}$</td>
<td>$\Delta \theta = \frac{1}{2} T^2$</td>
<td>$T \xrightarrow{\Delta \theta} \frac{1}{2} T^2$</td>
</tr>
<tr>
<td></td>
<td>Fluid capacitance</td>
<td>$Q = C_i \frac{dE}{dt}$</td>
<td>$E = \frac{1}{2} C_i Q^2$</td>
<td>$Q \xrightarrow{E} \frac{1}{2} C_i Q^2$</td>
</tr>
<tr>
<td></td>
<td>Thermal capacitance</td>
<td>$q = C_T \frac{dE}{dt}$</td>
<td>$E = C_T q$</td>
<td>$q \xrightarrow{E} C_T q$</td>
</tr>
<tr>
<td>Energy dissipators</td>
<td>Electrical resistance</td>
<td>$i = \frac{1}{R} \frac{dE}{dt}$</td>
<td>$E = \frac{1}{2} R i^2$</td>
<td>$i \xrightarrow{E} \frac{1}{2} R i^2$</td>
</tr>
<tr>
<td></td>
<td>Translational damper</td>
<td>$F = b \frac{d\Delta x}{dt}$</td>
<td>$\Delta x = \frac{1}{2} F^2$</td>
<td>$F \xrightarrow{\Delta x} \frac{1}{2} F^2$</td>
</tr>
<tr>
<td></td>
<td>Rotational damper</td>
<td>$T = b \frac{d\Delta \theta}{dt}$</td>
<td>$\Delta \theta = \frac{1}{2} T^2$</td>
<td>$T \xrightarrow{\Delta \theta} \frac{1}{2} T^2$</td>
</tr>
<tr>
<td></td>
<td>Fluid resistance</td>
<td>$Q = \frac{1}{R_c} P_2$</td>
<td>$P_2 = \frac{1}{2} R_c Q^2$</td>
<td>$Q \xrightarrow{P_2} \frac{1}{2} R_c Q^2$</td>
</tr>
<tr>
<td></td>
<td>Thermal resistance</td>
<td>$q = \frac{1}{R_T} \frac{dE}{dt}$</td>
<td>$E = \frac{1}{2} R_T q^2$</td>
<td>$q \xrightarrow{E} \frac{1}{2} R_T q^2$</td>
</tr>
</tbody>
</table>

Source: Dorf & Bishop, Modern Control Systems, 12th Ed., p. 74
Next Time…

- We’ll talk about System Models

- Review:
  - Phasers, complex numbers, polar to rectangular, and general functional forms.
  - Chapter 1 of Lathi (particularly the first sections on signals & classification thereof)

- Register on Platypus

- Try the practise assignment (will be posted soon)