Frequency Response & Filter Analysis

ELEC 3004: Digital Linear Systems Signals & Controls
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(with material from Kumaresan, Continuous-Time Fourier Transform, URI)

Lecture 5

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Typical Linear Processors

- **Convolution**: \( h(n,k)=h(n-k) \)
- **Cross Correlation**: \( h(n,k)=h(n+k) \)
- **Auto Correlation**: \( h(n,k)=x(k-n) \)
- **Cosine Transform**: \( h(n,k)=\cos\left(\frac{2\pi}{N}nk\right) \)
- **Sine Transform**: \( h(n,k)=\sin\left(\frac{2\pi}{N}nk\right) \)
- **Fourier Transform**: \( h(n,k)=\exp\left(j\frac{2\pi}{N}nk\right) \)
Transform Analysis

- Signal measured (or known) as a function of an independent variable
  - e.g., time: \( y = f(t) \)
- However, this independent variable may not be the most appropriate/informative
  - e.g., frequency: \( Y = f(w) \)
- Therefore, need to transform from one domain to the other
  - e.g., time \( \leftrightarrow \) frequency
  - As used by the human ear (and eye)

Signal processing uses Fourier, Laplace, & \( z \) transforms etc

---

Sinusoids and Linear Systems

\[
x(t) \text{ or } x(n) \quad \rightarrow \quad h(t) \text{ or } h(n) \quad \rightarrow \quad y(t) \text{ or } y(n)
\]

If \( x(t) = A \cos(\omega_0 t + \theta_0) \)

or \( x(n) = A \cos(\omega_0 n t + \theta_0) \)

then in steady state

\[
y(t) = AC(\omega_0) \cos(\omega_0 t + \theta_0 + \theta(\omega_0))
\]

\[
y(n) = AC(\omega_0 T) \cos(\omega_0 n t + \theta_0 + \theta(\omega_0 T))
\]
Sinusoids and Linear Systems

- The pair of numbers $C(w_0)$ and $q(w_0)$ are the complex gain of the system at the frequency $w_0$.
- They are respectively, the magnitude response and the phase response at the frequency $w_0$.

$$y(t) = AC(\omega_0) \cos(\omega_0 t + \theta_0 + \theta(\omega_0))$$
$$y(n) = AC(\omega_0 T) \cos(\omega_0 nt + \theta_0 + \theta(\omega_0 T))$$

Why Use Sinusoids?

- Why probe system with sinusoids?
- Sinusoids are eigenfunctions of linear systems???
- What the hell does that mean?
- Sinusoid in implies sinusoid out
- Only need to know phase and magnitude (two parameters) to fully describe output rather than whole waveform
  - $\sin + \sin = \sin$
  - derivative of $\sin = \sin$ (phase shifted - $\cos$)
  - integral of $\sin = \sin$ (-$\cos$)
- Sinusoids maintain orthogonality after sampling (not true of most orthogonal sets)
Frequency Response

**Fourier Series** → Fourier Transforms

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**Fourier Series**

- Deal with continuous-time periodic signals.
- Discrete frequency spectra.

A Periodic Signal

Source: URI ELE436
Two Forms for Fourier Series

Sinusoidal Form

\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi nt}{T} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi nt}{T} \right) \]

- \[ a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \, dt \]
- \[ a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t) \, dt \]
- \[ b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega_0 t) \, dt \]

Complex Form:

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \]

\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} \, dt \]

Source: URI ELE436

Fourier Series

- Any finite power, periodic, signal \( x(t) \)
  - period \( T \)
- can be represented as \((\infty)\) summation of
  - sine and cosine waves
- Called: Trigonometrical Fourier Series

\[ x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t) \]

- Fundamental frequency \( \omega_0 = 2\pi/T \text{ rad/s} \) or \( 1/T \text{ Hz} \)
- DC (average) value \( A_0/2 \)
Frequency representation (spectrum) shows signal contains:
- 2Hz and 5Hz components (sinewaves) of equal amplitude.

Fourier Series Coefficients
- An & Bn calculated from the signal, x(t)
  - called: Fourier coefficients

\[
A_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(nw_0 t) dt \quad n = 0,1,2,\cdots
\]
\[
B_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(nw_0 t) dt \quad n = 1,2,3,\cdots
\]

Note: Limits of integration can vary, provided they cover one period.
Fourier Series Coefficients

- Approximation with 1st, 3rd, 5th, & 7th Harmonics added, note:
  - ‘Ringing’ on edges due to series truncation
  - Often referred to as Gibb’s phenomenon
- Fourier series converges to original signal if
  - Dirichlet conditions satisfied
  - Closer approximation with more harmonics

Example: Square wave

\[
x(t) = \begin{cases} 
  1, & 0 < t < 1; \\
  -1, & 1 < t < 2; \\
  x(t + 2), & \text{periodic! i.e., } x(t + 2) = x(t)
\end{cases}
\]

\[
A_n = \frac{2}{\pi} \int_0^1 x(t) \cos(n\pi t) dt = \frac{1}{\pi} \int_0^1 \cos(n\pi t) dt - \frac{2}{\pi} \cos(n\pi) dt
\]

\[
A_n = \left[ -\sin(n\pi t) \right]_0^1 - \left[ -\sin(n\pi t) \right]_1^2 = 0
\]

No cos terms as \( \sin(n\pi) = 0 \quad \forall \ n \)

\[
x(t) \text{ has odd symmetry}
\]

\[
B_n = \frac{2}{\pi} \int_0^1 x(t) \sin(n\pi t) dt = \frac{1}{\pi} \int_0^1 \sin(n\pi t) dt - \frac{2}{\pi} \sin(n\pi) dt
\]

\[
B_n = \left[ -\cos(n\pi t) \right]_0^1 - \left[ -\cos(n\pi t) \right]_1^2 = -\frac{\cos(n\pi) + 1}{n\pi} + \frac{1}{n\pi} - \frac{\cos(n\pi)}{n\pi}
\]

\[
B_n = \frac{2}{n\pi} (1 - \cos(n\pi)) \quad \text{Sin terms only}
\]
Example: Square wave

Therefore, Trigonometric Fourier series is,

\[ x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - \cos(n\pi)) \sin(n\pi t) \]

Expanding the terms gives,

\[ x(t) = \frac{4}{\pi} \sin(\pi t) \quad \text{(fundamental)} \]
\[ + 0 \quad \text{(second harmonic)} \]
\[ + \frac{4}{3\pi} \sin(3\pi t) \quad \text{(third harmonic)} \]
\[ + 0 \quad \text{(fourth harmonic)} \]
\[ + \frac{4}{5\pi} \sin(5\pi t) \quad \text{(fifth harmonic)} \]
\[ + \text{etc} \]

- Only odd harmonics;
- In proportion \(1, 1/3, 1/5, 1/7, \ldots\);
- Higher harmonics contribute less;
- Therefore, converges

How to Deal with Aperiodic Signal?

A Periodic Signal

\[ f(t) \]
\[ T \]
\[ t \]

If \( T \to \infty \), what happens?

Source: URI ELE436
Fourier Integral

\[ f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 n} \]
\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_0 t} dt \]
\[ = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{T} \int_{-T/2}^{T/2} f_T(\tau) e^{-j\omega_0 \tau} d\tau \right] e^{j\omega_0 n} \]
\[ = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \int_{-T/2}^{T/2} f_T(\tau) e^{-j\omega_0 \tau} d\tau \right] \omega_0 e^{j\omega_0 n} \]
\[ = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \int_{-T/2}^{T/2} f_T(\tau) e^{-j\omega_0 \tau} d\tau \right] e^{j\omega_0 \Delta \omega} \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f_T(\tau) e^{-j\omega_0 \tau} d\tau \right] e^{j\omega_0 \Delta \omega} d\omega \]

Let \( \Delta \omega = \omega_0 = \frac{2\pi}{T} \)

\( T \to \infty \Rightarrow d\omega = \Delta \omega \approx 0 \)

Source: URI ELE436

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Fourier Integral

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} d\tau \right] e^{j\omega t} d\omega \]

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \quad \text{Synthesis} \]
\[ F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad \text{Analysis} \]

Source: URI ELE436
## Fourier Series vs. Fourier Integral

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<td>$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}$</td>
<td>$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jnw_0 t} , dt$</td>
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<tr>
<td>Fourier Integral:</td>
<td>Non-Period Function</td>
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<tr>
<td>$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} , d\omega$</td>
<td>$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} , dt$</td>
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Source: URI ELE436

## Complex Fourier Series (CFS)

- Also called Exponential Fourier series
  - As it uses Euler’s relation
    $$A \exp(jw_0 t) = A \cos(w_0 t) + jA \sin(w_0 t)$$
    which implies,
    $$\cos(nw_0 t) = \frac{\exp(jnw_0 t) + \exp(-jn w_0 t)}{2}$$
    $$\sin(nw_0 t) = \frac{\exp(jnw_0 t) - \exp(-jn w_0 t)}{2j}$$

- FS as a Complex phasor summation
  $$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn w_0 t)$$
  Where $X_n$ are the CFS coefficients
Complex Fourier Coefficients

- Again, Xₙ calculated from x(t)
- Only one set of coefficients, Xₙ
  - but, generally they are complex

\[ Xₙ = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \exp(-jnw₀t) dt \]

Remember: fundamental w₀ = 2π/T!

Relationships

- There is a simple relationship between
  - trigonometrical and
  - complex Fourier coefficients,

\[
X₀ = \frac{A₀}{2}
\]

\[
Xₙ = \begin{cases} 
\frac{Aₙ - jBₙ}{2}, & n > 0; \\
\frac{Aₙ + jBₙ}{2}, & n < 0.
\end{cases}
\]

Constrained to be symmetrical, i.e., complex conjugate

\[ X_{-n} = Xₙ^* \]

Therefore, can calculate simplest form and convert
Example: Complex FS

- Consider the pulse train signal

\[
x(t) = \begin{cases} 
  A, & 0 \leq |\tau| \leq \frac{T}{2}; \\
  0, & \frac{T}{2} < |\tau| \leq T; \\
  x(t+T). & 
\end{cases}
\]

- Has complex Fourier series:

\[
X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \exp(-j n \omega_0 \tau) dt = \frac{1}{T} \int_{-T/2}^{T/2} A \exp(-j n \omega_0 \tau) dt 
\]

Note: \( \tau \) by \( \tau/\tau \) ...

\[
= \frac{-A \tau}{jn \omega_0 T} \left[ \exp \left( -j n \omega_0 \frac{\tau}{2} \right) - \exp \left( j n \omega_0 \frac{\tau}{2} \right) \right] 
\]

Note: \( n \) is the ind. variable

Example: Complex FS

- Which using Euler’s identity reduces to:

\[
X_n = \frac{A \tau}{T} \frac{\sin(n \omega_0 \tau/2)}{n \omega_0 \tau/2} = \frac{A \tau}{T} \text{sa}(n \omega_0 \tau/2) 
\]

\[
\omega_0 = \frac{2\pi}{T} 
\]

Note: letting \( \theta = \frac{n \omega_0 \tau}{2} \)

\[
\exp(-j \theta) - \exp(j \theta) 
\]

\[
= \cos(-\theta) + j \sin(-\theta) - (\cos(\theta) + j \sin(\theta)) 
\]

\[
= \cos(\theta) - j \sin(\theta) - \cos(\theta) - j \sin(\theta) = -2 j \sin(\theta) 
\]
Dirichlet Conditions

For Fourier series to converge, f(t) must be:

- defined & single valued
- continuous and have a finite number of finite discontinuities within a periodic interval, and
- piecewise continuous in periodic interval, as must $f'(t)$ be absolutely integrable; i.e.,
  - i.e., have finite energy
- have a finite number of finite discontinuities within a finite interval, and
- have a finite number of maxima and minima within a finite interval

\[ \int_{-\infty}^{\infty} |f(t)| \, dt < \infty \]

Note: Periodic signals have FT, if we use impulse functions, $\delta(w)$

Frequency Response

Fourier Series $\rightarrow$ Fourier Transforms
**Fourier Transform**

- A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

1-D Example:

- When you let these three waves interfere with each other you get your original wave function!

Source: Tufts Uni Sykes Group

**Fourier Series**

- What we have produced is a processor to calculate one coefficient of the complex Fourier Series
- Fourier Series Coefficients = Heterodyne and average over observation interval $T$

$$C_k = \frac{1}{T} \int_{0}^{T} h(t) e^{-j \frac{2\pi}{T}kt} dt$$
Fourier Transform

- If we change the limits of integration to the entire real line, remove the division by T, and make the frequency variable continuous, we get the Fourier Transform

\[ C(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} \, dt \]

Fourier Transform (is not the Fourier Series per se)

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Source: URI ELE436
Fourier Transform

- Fourier series
  - Only applicable to periodic signals

- Real world signals are rarely periodic

- Develop Fourier transform by
  - Examining a periodic signal
  - Extending the period to infinity

Problem: as $T \to \infty$, $X_n \to 0$
  - i.e., Fourier coefficients vanish!

Solution: re-define coefficients
  - $X_n' = T \times X_n$

As $T \to \infty$
  - (harmonic frequency) $nw_0 \to w$ (continuous freq.)
  - (discrete spectrum) $X_n' \to X(w)$ (continuous spect.)
  - $w_0$ (fundamental freq.) reduces $\to dw$ (differential)
    - Summation becomes integration
Fourier Transform Pair

**Inverse Fourier Transform:**

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \]

**Synthesis**

**Fourier Transform:**

\[ F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \]

**Analysis**

---

Continuous Spectra

\[ F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \]

\[ F(j\omega) = F_R(j\omega) + jF_I(j\omega) \]

\[ = |F(j\omega)| e^{j\phi(\omega)} \]

Magnitude

Phase

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Source: URI ELE436
Time limited

\[ x(t) \text{ with } \tau = 1 \]

\[ X(w) = \text{sinc}(w/2\pi) \]

Infinite bandwidth

\[ x(t) \text{ with } \tau = 2 \]

\[ X(w) = 2 \text{sinc}(w/\pi) \]

Parseval's Theorem
Pulse width $\tau = 4$

$\text{rect}(t/4)$

$\text{rect}(t/4)$

$4 \text{sinc}(2w/\pi)$

$4 \text{sinc}(2w/\pi)$

$\text{rect}(t/8)$

$\text{rect}(t/8)$

$8 \text{sinc}(4w/\pi)$

$8 \text{sinc}(4w/\pi)$
Properties of Fourier Transform

- **Linearity**
  - $F\{a \cdot x(t) + b \cdot y(t)\} = a \cdot X(w) + b \cdot Y(w)$

- **Time and frequency scaling**
  - $F\{x(at)\} = \frac{1}{a} X(w/a)$
  - broader in time $\Rightarrow$ narrower in frequency
    - and vice versa

- **Symmetry (duality)**
  - $2\pi x(-w) = \int X(t) \exp(-jwt) dt$
    - i.e., Fourier transform ‘pairs’

Time limited signal limited has infinite bandwidth;
Signal of finite bandwidth has infinite time support
Properties of Fourier Transform

- if...
- $x(t)$ is real
- Then...
- $X(-w) = X(w)^*$
  - $\Re \{X(w)\}$ is even
  - $\Im \{X(w)\}$ is odd
  - $|X(w)|$ is even
  - $\angle X(w)$ is odd

- $x(t)$ is real and even
- $X(w)$ is real and even
- $x(t)$ is real and odd
- $X(w)$ is imaginary and odd

Fourier Transforms

- $X(\omega) = \delta(\omega - \omega_0)$
- $x(t) = \mathcal{F}^{-1}\{X(\omega)\}$
- $x(t) = \frac{1}{2\pi} \exp(j\omega_0 t)$

Note: $\cos(\omega_0 t)$ has $\infty$ energy! But is dual of $\delta(w - \omega_0)$

$x(t) = \cos(\omega_0 t)$
(real & even)

$X(w) = \pi[\delta(w - \omega_0) + \delta(w + \omega_0)]$
(real and even)
Fourier Transforms

Note: sin & cos have same Mag spectrum
Phase is only difference

\[ x(t) = \sin(w_0 t) \]
(real and odd)

\[ X(w) = j\pi [\delta(w+w_0) - \delta(w-w_0)] \]
(imaginary & odd)

Properties of Fourier Transform

- **Time Shift**
  - \[ F \{x(t - \alpha)\} = \exp(-j\alpha w)X(w) \]
    - time shift \(\Rightarrow\) phase shift

- **Convolution and multiplication**
  - \[ F \{x(t) * y(t)\} = X(w) \cdot Y(w) \]
    - i.e., implement convolution in Fourier domain
  - \[ F \{x(t) \cdot y(t)\} = 1/2\pi \{X(w) * Y(w)\} \]
    - i.e., Fourier interpretation of multiplication (e.g., frequency modulation)
Modifies phase only

as 
\[
\cos^2 + \sin^2 = 1
\]
More properties of the FT

- Differentiation in time
  \[ F \left\{ \frac{d^n}{dt^n} x(t) \right\} = (j\omega)^n X(\omega) \]
  Differentiation \( \Rightarrow \times \omega \)
  (Note: HPF & DC x zero)

- Integration in time
  \[ F \left\{ \int_{-\infty}^{t} x(t) dt \right\} = \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega) \]
  Integration \( \Rightarrow /\omega + \text{DC offset} \) (LPF & opposite of differentiation)

More Fourier Transforms

See Tutorial 2 for proof…

Impulse train, ‘comb’ or ‘Shah’ function
More Fourier Transforms

Limit of previous as $\Delta t \to \infty$ and $\Delta t \to 0$ respectively
Note: $f(t) = 1$ has $\infty$ energy! But is dual of $\delta(t)$

\[
f(t) = \delta(t) \quad \Longrightarrow \quad F(w) = 1
\]

\[
f(t) = 1 \quad \Longrightarrow \quad F(w) = 2\pi\delta(w)
\]

Note: $u(t)$ also has $\infty$ energy! But $F\{u(t)\} = F\{\delta(t)\}$ i.e., apply integration property

Interpretation of Fourier Transform

- Represents (usually finite energy) signals
  - as sum of cosine waves
    - at all possible frequencies
    - $|X(w)|dw/2\pi$ is amplitude of cosine wave
      - i.e., in frequency band $w$ to $w + dw$
    - $\angle X(w)$ is phase shift of cosine wave
- Also represents finite power, periodic signals
  - Using $\delta(w)$
- Distribution with frequency of
  - both magnitude & phase
  - called a Frequency spectrum (continuous)
Negative Frequency

- Q: What is negative frequency?
- A: A mathematical convenience
- Trigonometrical FS
  - periodic signal is made up from
  - sum 0 to $\infty$ of sine and cosines ‘harmonics’
- Complex FS and the FT
  - use $\exp(\pm j\omega t)$ instead of $\cos(\omega t)$ and $\sin(\omega t)$
  - signal is sum from 0 to $\infty$ of $\exp(\pm j\omega t)$
  - same as sum $-\infty$ to $\infty$ of $\exp(-j\omega t)$
  - which is more compact (i.e., less chalk!)

\[
Ae^{jwt} = A(\cos(\omega t) + j\sin(\omega t))
\]

+ve frequency

\[
Ae^{-jwt} = A\cos(-\omega t) + jA\sin(-\omega t)
\]

-ve frequency
Fourier Image Examples

Lena

Bridge

Fourier Magnitude and Phase

Bridge spectra look similar

20*log10(abs(fft(Lena)))

angle(fft(Lena))

'random' range(±π)
Magnitude and Phase Only

\[
\text{ifft}(\text{abs}(\text{fft}(\text{Lena})) + \text{angle}(0)) \quad \text{ifft}(\text{abs}(\text{fft}(\text{Bridge})) + \text{angle}(\text{fft}(\text{Lena})))
\]

Lena magnitude only \quad Lena phase + bridge magnitude

Note: titles are illustrative only and are not the actual Matlab commands used!

Questions

- If \( F\{x(t)\} = X(w) \)
  - \( F\{x(2t)\} = ? \)
  - \( F\{x(t/4)\} = ? \)
- \( F\{\delta(t)\} = ? \)
- \( F\{1\} = ? \)
Questions

- If $F\{x(t)\} = X(w)$
  - $F\{x(2t)\} = 1/2X(w/2)$
    - narrower in $t$ $\Leftrightarrow$ broader in freq
  - $F\{x(t/4)\} = 4X(4w)$
    - broader in $t$ $\Leftrightarrow$ narrower in freq (but increased amplitude)

- $F\{\delta(t)\} = 1$
  - i.e. flat spectrum (all frequencies equally)

- $F\{1\} = \delta(w)$
  - i.e. impulse at DC only

Frequency

- How often the signal repeats
- Can be analyzed through Fourier Transform

Examples:

- Signals in time domain
- Fourier Transform
- Signals in frequency domain
Noise

Various Types:
- Thermal (white):
  - Johnson noise, from thermal energy inherent in mass.

- Flicker or 1/f noise:
  - Pink noise
  - More noise at lower frequency

- Shot noise:
  - Noise from quantum effects as current flows across a semiconductor barrier

- Avalanche noise:
  - Noise from junction at breakdown (circuit at discharge)
How to beat the noise

- Filtering (Narrow-banding): Only look at particular portion of frequency space
- Multiple measurements …
- Other (modulation, etc.) …

Noise ⊆ Uncertainty

- **Uncertainty:**
  All measurement has some approximation
  A. **Statistical uncertainty:** quantified by mean & variance
  B. **Systematic uncertainty:** non-random error sources

- **Law of Propagation of Uncertainty**
  - Combined uncertainty is root squared

\[ u_C = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} \]
1. **Over time:** multiple readings of a quantity over time
   - “stationary” or “ergodic” system
   - Sometimes called “integrating”

2. **Over space:** single measurement (summed) from multiple sensors each distributed in space

3. **Same Measurand:** multiple measurements take of the same observable quantity by multiple, related instruments
   - e.g., measure position & velocity simultaneously
   - Basic “sensor fusion”

\[
\sigma_{\text{final}} = \left[ \sigma_1^{-1} + \sigma_2^{-1} + \cdots + \sigma_n^{-1} \right]^{-1}
\]

---

**Multiple Measurements Example**

- What time was it when this picture was taken?
- What was the temperature in the room?
• **Frequency-shaping filters**: LTI systems that change the shape of the spectrum
• **Frequency-selective filters**: Systems that pass some frequencies undistorted and attenuate others

**Filters**

- **Lowpass**
- **Bandpass**
- **Highpass**
- **Bandstop (Notch)**

**Specified Values:**

- **$G_p$** = minimum passband gain
- Typically:
  \[ G_p = \frac{1}{\sqrt{2}} = -3dB \]
- **$G_s$** = maximum stopband gain
  - **Low**, not zero (sorry!)
  - For realizable filters, the gain cannot be zero over a finite band (Paley-Wiener condition)

- **Transition Band**: transition from the passband to the stopband $\Rightarrow \omega_p \neq \omega_s$
Filter Design & z-Transform

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Mapping</th>
<th>Design Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-pass</td>
<td>$z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$</td>
<td>$\alpha = \frac{\sin(\omega_c - \omega_c'/2)}{\sin(\omega_c + \omega_c'/2)}$, $\omega_c'$ = desired cutoff frequency</td>
</tr>
<tr>
<td>High-pass</td>
<td>$z^{-1} \rightarrow -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$</td>
<td>$\alpha = \frac{-\cos(\omega_c + \omega_c'/2)}{\cos(\omega_c - \omega_c'/2)}$, $\omega_c'$ = desired cutoff frequency</td>
</tr>
<tr>
<td>Bandpass</td>
<td>$z^{-1} \rightarrow -\frac{z^{-2} - [2\alpha]/{(\beta + 1)}z^{-1} + [(\beta - 1)/{(\beta + 1)}]}{[(\beta - 1)/{(\beta + 1)}]z^{-2} - [2\alpha]/{(\beta + 1)}z^{-1} + 1}$</td>
<td>$\alpha = \frac{\cos(\omega_{c2} + \omega_{c1}/2)}{\cos(\omega_{c2} - \omega_{c1}/2)}$, $\beta = \sec(\omega_{c2} - \omega_{c1}/2)\tan(\omega_{c2}/2)$, $\omega_{c1}$ = desired lower cutoff frequency, $\omega_{c2}$ = desired upper cutoff frequency</td>
</tr>
<tr>
<td>Bandstop</td>
<td>$z^{-1} \rightarrow -\frac{z^{-2} - [2\alpha]/{(\beta + 1)}z^{-1} + [(1 - \beta)/(1 + \beta)]}{[(1 - \beta)/(1 + \beta)]z^{-2} - [2\alpha]/{(\beta + 1)}z^{-1} + 1}$</td>
<td>$\alpha = \frac{\cos(\omega_{c1} + \omega_{c2}/2)}{\cos(\omega_{c1} - \omega_{c2}/2)}$, $\beta = \sec(\omega_{c1} - \omega_{c1}/2)\tan(\omega_{c1}/2)$, $\omega_{c1}$ = desired lower cutoff frequency, $\omega_{c2}$ = desired upper cutoff frequency</td>
</tr>
</tbody>
</table>

Butterworth Filters

- Butterworth: Smooth in the pass-band
- The amplitude response $|H(j\omega)|$ of an $n^{th}$ order Butterworth low pass filter is given by:

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

- The normalized case ($\omega_c = 1$)

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}}$$

$H(j\omega)H(-j\omega) = |H(j\omega)|^2 = \frac{1}{1 + \omega^{2n}}$

Recall that: $|H(j\omega)|^2 = H(j\omega)H(-j\omega)$
Butterworth Filters

Increasing the order, increases the number of poles:

- Odd orders (n=1,3,5…):
  - Have a pole on the Real Axis

- Even orders (n=2,4,6…):
  - Have a pole on the off axis

Butterworth Filters of Increasing Order: Seeing this Using a Pole-Zero Diagram

- Increasing the order, increases the number of poles:

  ➔ Odd orders (n=1,3,5…):
  - Have a pole on the Real Axis

  ➔ Even orders (n=2,4,6…):
  - Have a pole on the off axis

Angle between poles: $\frac{\pi}{n}$
Butterworth Filters: Pole-Zero Diagram

- Since $H(s)$ is stable and causal, its poles must lie in the LHP
- Poles of $-H(s)$ are those in the RHP
- Poles lie on the unit circle (for a normalized filter)

$$H(s) = \frac{1}{(s - s_1)(s - s_2)\ldots(s - s_n)}$$

where:

$$s_k = e^{j\frac{\pi}{2n}(2k+n-1)} = \cos\frac{\pi}{2n}(2k+n-1) + j\sin\frac{\pi}{2n}(2k+n-1) \quad k = 1, 2, 3, \ldots, n$$

$n$ is the order of the filter

Butterworth Filters: 4th Order Filter Example

- Plugging in for $n=4$, $k=1,\ldots,4$:

$$H(s) = \frac{1}{(s + 0.3827 - j0.9239)(s + 0.3827 + j0.9239)(s + 0.9239 - j0.3827)(s + 0.9239 + j0.3827)}$$

$$= \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}$$

$$= \frac{1}{s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1}$$

- We can generalize $\Rightarrow$ Butterworth Table

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.41421356</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.00000000</td>
<td>2.00000000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.61312593</td>
<td>3.41421356</td>
<td>2.61312593</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3.2366798</td>
<td>5.2366798</td>
<td>5.2366798</td>
<td>3.2366798</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.86370331</td>
<td>7.46410162</td>
<td>9.14162017</td>
<td>7.46410162</td>
<td>3.86370331</td>
</tr>
</tbody>
</table>

This is for 3dB bandwidth at $\omega_c=1$
Butterworth Filters: Scaling Back (from Normalized)
• Start with Normalized equation & Table
• Replace $\omega$ with $\omega/\omega_c$ in the filter equation

• For example:
  for $f_c=100$Hz $\rightarrow \omega_c=200\pi$ rad/sec

  From the Butterworth table: for $n=2$, $a_1=\sqrt{2}$
  Thus:

  $$H(s) = \frac{1}{\left(\frac{s}{200\pi}\right)^2 + \sqrt{2}\left(\frac{s}{200\pi}\right) + 1}$$

  $$= \frac{1}{s^2 + 200\pi \sqrt{2} + 40,000\pi^2}$$

Butterworth: Determination of Filter Order
• Define $G_x$ as the gain of a lowpass Butterworth filter at $\omega = \omega_x$
• Then:

  $$\hat{G}_p = -10 \log \left[ 1 + \left( \frac{\omega_p}{\omega_c} \right)^{2n} \right]$$

  $$\hat{G}_s = -10 \log \left[ 1 + \left( \frac{\omega_s}{\omega_c} \right)^{2n} \right]$$

  And thus:

  $$\hat{G}_s = -10 \log \left[ 1 + \left( \frac{\omega_s}{\omega_c} \right)^{2n} \right]$$

  Or alternatively:

  $$\omega_c = \frac{\omega_p}{\left[ 10^{-\hat{G}_s/10} - 1 \right]^{1/2n}}$$

  $$\omega_c = \frac{\omega_s}{\left[ 10^{-\hat{G}_p/10} - 1 \right]^{1/2n}}$$

  Solving for $n$ gives:

  $$n = \frac{\log \left[ \left( 10^{-\hat{G}_s/10} - 1 \right) / \left( 10^{-\hat{G}_p/10} - 1 \right) \right]}{2 \log (\omega_s/\omega_p)}$$

**PS.** See Lathi 4.10 (p. 453) for an example in MATLAB
Chebyshev Filters

- **equal-ripple:** Because all the ripples in the passband are of equal height.
- If we reduce the ripple, the passband behaviour improves, but it does so at the cost of stopband behaviour.

Chebyshev Filters

- Chebyshev Filters: Provide tighter transition bands (sharper cutoff) than the same-order Butterworth filter, but this is achieved at the expense of inferior passband behavior (rippling).
- For the lowpass (LP) case: at higher frequencies (in the stopband), the Chebyshev filter gain is smaller than the comparable Butterworth filter gain by about $6(n - 1) \text{ dB}$.

- The amplitude response of a normalized Chebyshev lowpass filter is:

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 C_n^2(\omega)}}$$

Where $C_n(\omega)$, the nth-order Chebyshev polynomial, is given by:

$$C_n(\omega) = \cos(n \cos^{-1} \omega)$$

$$C_n(\omega) = \cosh(n \cosh^{-1} \omega)$$

and where $C_n$ is given by:

<table>
<thead>
<tr>
<th>n</th>
<th>$C_n(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\omega$</td>
</tr>
<tr>
<td>2</td>
<td>$2\omega^2 - 1$</td>
</tr>
<tr>
<td>3</td>
<td>$4\omega^3 - 3\omega$</td>
</tr>
<tr>
<td>4</td>
<td>$8\omega^4 - 8\omega^2 + 1$</td>
</tr>
<tr>
<td>5</td>
<td>$16\omega^5 - 20\omega^3 + 5\omega$</td>
</tr>
<tr>
<td>6</td>
<td>$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$</td>
</tr>
</tbody>
</table>
Normalized Chebyshev Properties

- It’s normalized: The passband is $0 < \omega < 1$
- **Amplitude response**: has ripples in the passband and is smooth (monotonic) in the stopband
- **Number of ripples**: there is a total of $n$ maxima and minima over the passband $0 < \omega < 1$

\[
C_n^2(0) = \begin{cases} 
0, & n : \text{odd} \\
1, & n : \text{even}
\end{cases}
\Rightarrow |H(0)| = \begin{cases} 
1, & n : \text{odd} \\
\frac{1}{\sqrt{1+c^2}}, & n : \text{even}
\end{cases}
\]

- $\epsilon$: ripple height $\Rightarrow r = \sqrt{1 + \epsilon^2}$
- The Amplitude at $\omega=1$: $\frac{1}{r} = \frac{1}{\sqrt{1+c^2}}$
- For Chebyshev filters, the ripple $r$ dB takes the place of $G_p$

Determination of Filter Order

- The gain is given by: $\hat{G} = -10 \log \left[ 1 + \epsilon^2 C_n^2(\omega) \right]$
  Thus, the gain at $\omega_s$ is: $\epsilon^2 C_n^2(\omega_s) = 10^{-\hat{G}/10} - 1$

- Solving:
  \[
n = \frac{1}{\cosh^{-1}(\omega_p)} \cosh^{-1} \left[ \frac{10^{-\hat{G}/10} - 1}{10^{\hat{G}/10} - 1} \right]^{1/2}
\]

- General Case:
  \[
n = \frac{1}{\cosh^{-1}(\omega_s/\omega_p)} \cosh^{-1} \left[ \frac{10^{-\hat{G}/10} - 1}{10^{\hat{G}/10} - 1} \right]^{1/2}
\]
Chebyshev Pole Zero Diagram

- Whereas **Butterworth** poles lie on a semi-circle, The poles of an \( n \)th-order normalized **Chebyshev** filter lie on a semiellipse of the major and minor semiaxes:

\[
a = \sinh \left( \frac{1}{n} \sinh^{-1} \left( \frac{1}{\epsilon} \right) \right) \quad \& \quad b = \cosh \left( \frac{1}{n} \sinh^{-1} \left( \frac{1}{\epsilon} \right) \right)
\]

And the poles are at the locations:

\[
H(s) = \frac{1}{(s - s_1)(s - s_2) \ldots (s - s_n)}
\]

\[
s_k = -\sin \left[ \frac{(2k - 1)\pi}{2n} \right] \sinh x + j \cos \left[ \frac{(2k - 1)\pi}{2n} \right] \cosh x, \ k = 1, \ldots, n
\]

Ex: Chebyshev Pole Zero Diagram for \( n=3 \)

Procedure:

1. Draw two semicircles of radii \( a \) and \( b \) (from the previous slide).
2. Draw radial lines along the corresponding Butterworth angles \((\pi/n)\) and locate the \( n \)th-order Butterworth poles (shown by crosses) on the two circles.
3. The location of the \( k \)th Chebyshev pole is the intersection of the horizontal projection and the vertical projection from the corresponding \( k \)th Butterworth poles on the outer and the inner circle, respectively.
Chebyshev Values / Table

\[ \mathcal{H}(s) = \frac{K_n}{C'(s)} = \frac{K_n}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \]

\[ K_n = \begin{cases} 
    a_0 & \text{n odd} \\
    \frac{a_0}{\sqrt{1 + \epsilon^2}} = \frac{a_0}{10^{p/20}} & \text{n even}
\end{cases} \]

<table>
<thead>
<tr>
<th>n</th>
<th>a₀</th>
<th>a₁</th>
<th>a₂</th>
<th>a₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.9652267</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.1025103</td>
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<td></td>
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<td>3</td>
<td>0.4913067</td>
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<td>0.9883412</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.2756276</td>
<td>0.7426194</td>
<td>1.4539248</td>
<td>0.9528114</td>
</tr>
</tbody>
</table>

1 db ripple  \( (\epsilon = 1) \)

Other Filter Types:

**Chebyshev Type II = Inverse Chebyshev Filters**

- Chebyshev filters passband has ripples and the stopband is smooth.
- **Instead:** this has passband have smooth response and ripples in the stopband.
- Exhibits maximally flat passband response and equi-ripple stopband
- **Cheby2 in MATLAB**

\[ |\mathcal{H}(\omega)|^2 = 1 - |\mathcal{H}(1/\omega)|^2 = \frac{\epsilon^2C_n^2(1/\omega)}{1 + \epsilon^2C_n^2(1/\omega)} \]

Where: \( H \) is the Chebyshev filter system from before

- Passband behavior, especially for small \( \omega \), is **better** than Chebyshev
- Smallest transition band of the 3 filters (Butter, Cheby, Cheby2)
- Less time-delay (or phase loss) than that of the Chebyshev
- Both needs the **same order** \( n \) to meet a set of specifications.
- $$ (or number of elements): Cheby < Inverse \text{ Chebyshev} < \text{Butterworth} \) (of the same performance [not order])
Other Filter Types: 
**Elliptic Filters (or Cauer) Filters**

- Allow **ripple** in **both** the passband and the stopband, 
  - we can achieve **tighter** transition band

\[
|H(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 R_n^2(\omega)}},
\]

**Where:**  
- \( R_n \) is the \( n \)-th order Chebyshev rational function determined from a given ripple spec.  
- \( \epsilon \) controls the ripple  
  
- **Most efficient** (\( \eta \))  
  - the **largest ratio** of the passband gain to stopband gain  
  - **or** for a given ratio of passband to stopband gain, it requires the **smallest transition band**

- **in MATLAB:** `ellipord` followed by `ellip`

---

**In Summary**

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Passband Ripple</th>
<th>Stopband Ripple</th>
<th>Transition Band</th>
<th>MATLAB Design Command</th>
</tr>
</thead>
<tbody>
<tr>
<td>Butterworth</td>
<td>No</td>
<td>No</td>
<td>Loose</td>
<td><code>butter</code></td>
</tr>
<tr>
<td>Chebyshev</td>
<td>Yes</td>
<td>No</td>
<td>Tight</td>
<td><code>cheby</code></td>
</tr>
<tr>
<td>Chebyshev Type II</td>
<td>No</td>
<td>Yes</td>
<td>Tight</td>
<td><code>cheby2</code></td>
</tr>
<tr>
<td>(Inverse Chebyshev)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elliptic</td>
<td>Yes</td>
<td>Yes</td>
<td>Tightest</td>
<td><code>ellip</code></td>
</tr>
</tbody>
</table>