Sampling & Data Acquisition & Antialiasing Filters

ELEC 3004: **Digital Linear Systems** Signals & Controls
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Lecture 3

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Then a System is a **MATRIX**

\[
\begin{bmatrix}
  y[1] \\
  y[2] \\
  \vdots \\
  y[M]
\end{bmatrix} = \begin{bmatrix}
  D_{11} & D_{12} & \cdots & D_{1N} \\
  D_{21} & D_{22} & \cdots & D_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  D_{M1} & D_{M2} & \cdots & D_{MN}
\end{bmatrix}
\begin{bmatrix}
  u[1] \\
  u[2] \\
  \vdots \\
  u[N]
\end{bmatrix}.
\]

\[
y[i] = \sum_j D_{ij} u[j].
\]
Recall From Last Time …
Classifications of Systems

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Causality:
Causal (physical or nonanticipative) systems

• Is one for which the output at any instant \( t_0 \) depends only on the value of the input \( x(t) \) for \( t \leq t_0 \). Ex:
\[
\begin{align*}
  u(t) &= x(t - 2) \Rightarrow \text{causal} \\
  u(t) &= x(t - 2) + x(t + 2) \Rightarrow \text{noncausal}
\end{align*}
\]

• A “real-time” system must be causals
  – How can it respond to future inputs?
• A prophetic system: knows future inputs and acts on it (now)
  – The output would begin before \( t_0 \)
• In some cases Noncausal maybe modelled as causal with delay
• Noncausal systems provide an upper bound on the performance of causal systems
Causality:
Looking at this from the output’s perspective…

- **Causal** = The output before some time \( t \) does not depend on the input after time \( t \).

Given: \( y(t) = F(u(t)) \)
For:
\( \bar{u}(t) = u(t), \forall 0 \leq t < T \) or \([0, T)\)

Then for a \( T > 0 \):
\[ \rightarrow \bar{y}(t) = y(t), \forall 0 \leq t < T \]

A system with a memory

- Where past history (or derivative states) are **relevant** in determining the response

Ex:
- RC circuit: Dynamical
  - Clearly a function of the “capacitor’s past” (initial state) and
  - Time! (charge / discharge)
- R circuit: is memoryless \( \neg \) the output of the system
  (recall \( V=IR \)) at some time \( t \) only depends on the input at time \( t \)

Lumped/Distributed
- Lumped: Parameter is constant through the process & can be treated as a “point” in space
- Distributed: System dimensions \( \neq \) small over signal
  - Ex: waveguides, antennas, microwave tubes, etc.
Linear Time Invariant

\[ u(t) \xrightarrow{\text{LTI}} y(t) = u(t) * h(t) = F(\delta(t)) \]

- Linear & Time-invariant (of course - tautology!)
- Impulse response: \( h(t) = F(\delta(t)) \)
- Why?
  - Since it is linear the output response \( y(t) \) to any input \( x(t) \) is:
    \[
    x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau \\
    y(t) = F\left[\int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau\right] = \int_{-\infty}^{\infty} x(\tau) F(\delta(t-\tau)) \, d\tau \\
    h(t-\tau) = F(\delta(t-\tau)) \\
    \Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau = x(t) * h(t) 
    \]

- The output of any continuous-time LTI system is the **convolution** of input \( u(t) \) with the impulse response \( F(\delta(t)) \) of the system.

Linear Dynamic [Differential] System

\( \equiv \) LTI systems for which the input & output are linear ODEs

\[
a_0y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}
\]

Laplace:

\[
a_0Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s) \\
A(s)Y(s) = B(s)X(s)
\]

- Total response = Zero-input response + Zero-state response

\begin{align*}
\text{Initial conditions} & \quad \text{External Input}
\end{align*}
Linear Systems and ODE’s

- Linear system described by differential equation

\[ a_0 \frac{dy}{dt} + a_1 \frac{d^2y}{dt^2} + \cdots + a_n \frac{d^ny}{dt^n} = b_0 \frac{dx}{dt} + b_1 \frac{d^2x}{dt^2} + \cdots + b_m \frac{d^mx}{dt^m} \]

- Which using Laplace Transforms can be written as

\[ a_0 Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s) \]

\[ A(s)Y(s) = B(s)X(s) \]

where \( A(s) \) and \( B(s) \) are polynomials in \( s \)

Unit Impulse Response

- \( \delta(t) \): Impulsive excitation
- \( h(t) \): characteristic mode terms

**Ex:**

Determine the unit impulse response \( h(t) \) for the system specified by the equation

\[ (s^2 + 3s + 2) y(t) = 2x(t) \]  

This is a second-order system \( \left(
\begin{array}{c}
 2 \\
 3 \\
 2 \\
\end{array}
\right) \) with the characteristic polynomial

\[ s^2 + 3s + 2 = (s + 1)(s + 2) \]  

The characteristic roots of this system are \( s = -1 \) and \( s = -2 \). Therefore

\[ \delta(t) = e^{-t} + e^{-2t} \]  

Differentiation of the equation yields

\[ \delta(t) = -e^{-t} - 2e^{-2t} \]  

The initial conditions are given by \( \delta(t) = 2 \) for \( t \rightarrow \infty \)

\[ \delta(t) = 1 \]  

Solving the simultaneous equations yields

\[ c_1 = 1 \quad \text{and} \quad c_2 = -1 \]  

Therefore

\[ \delta(t) = e^{-t} - e^{-2t} \]  

Moreover, according to Eq. (2.26), \( \Phi(2) = 2 \) so that

\[ F(\Phi_1(t)) = \Phi_2(t) = \delta(t) = e^{-t} + e^{-2t} \]  

Also, in this case, \( \Phi_0 = 0 \) (the second-order term is absent in \( F(t) \)). Therefore

\[ h(t) = \Phi(t) \Phi(t) \delta(t) = \left( e^{-t} + e^{-2t} \right) \delta(t) \]
System Models

- Various things – all the same!

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Circuits

\[
\frac{V_2(t)}{V_1(t)} = \frac{1}{R C s}
\]

\[
\frac{v_{out}}{v_{in}} = \frac{1}{C_1 C_2 R_1 R_2 s^2 + C_2 (R_1 + R_2) s + 1}
\]
Motors

5. DC motor, field-controlled, rotational actuator

\[ \theta(s) \overset{V_f(s)}{\rightarrow} \frac{K_m}{s(Js + b)(Ljs + R_f)} \]

7. AC motor, two-phase control field, rotational actuator

\[ \theta(s) \overset{V_c(s)}{\rightarrow} \frac{K_m}{s(rxs + 1)} \]
\[ \tau = J/(b - m) \]
\[ m = \text{slope of linearized torque-speed curve (normally negative)} \]

Mechanical Systems

15. Accelerometer, acceleration sensor

\[ x_a(t) = y(t) - x_{a0}(t), \]
\[ X_a(s) = -s^2 \]
\[ X_{a0}(s) = s^2 + (b/M)s + k/M \]

For low-frequency oscillations, where \( \omega < \omega_n \),

\[ \frac{X_a(j\omega)}{X_{a0}(j\omega)} = \frac{-\omega^2}{k/M} \]
Thermal Systems

16. Thermal heating system

\[ \mathcal{F}(s) = \frac{1}{C_s + (QS + 1/R_s)}, \] where

- \( \mathcal{F} = \mathcal{F}_a - \mathcal{F}_b = \) temperature difference due to thermal process
- \( C_s = \) thermal capacitance
- \( Q = \) fluid flow rate = constant
- \( S = \) specific heat of water
- \( R_s = \) thermal resistance of insulation
- \( q(s) = \) transform of rate of heat flow of heating element

First Order Systems

First order systems

\[ ay' + by = 0 \quad (with \ a \neq 0) \]

righthand side is zero:
- called autonomous system
- solution is called natural or unforced response

can be expressed as

\[ Ty' + y = 0 \quad \text{or} \quad y' + ry = 0 \]

where
- \( T = a/b \) is a time (units: seconds)
- \( r = b/a = 1/T \) is a rate (units: 1/sec)
First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T(sY(s) - y(0)) + Y(s) = 0$$

solve for $Y(s)$ (algebraic)

$$Y(s) = \frac{T y(0)}{sT + 1} = \frac{y(0)}{s + 1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, $y$ decays exponentially

- $T$ gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half ($0.693 = \log 2$)
- $4.6T$ gives time to decay by 0.01 ($4.6 = \log 100$)

if $T < 0$, $y$ grows exponentially

- $|T|$ gives time to grow by $e \approx 2.72$
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100
First Order Systems

Examples

simple RC circuit:

\[ R \quad C \quad v \]

+ \quad \text{circuit equation: } RCv' + v = 0

- \quad \text{solution: } v(t) = v(0)e^{-t/(RC)}

population dynamics:
- \( y(t) \) is population of some bacteria at time \( t \)
- growth (or decay if negative) rate is \( y' = by - dy \) where \( b \) is birth rate, \( d \) is death rate
- \( y(t) = y(0)e^{(b-d)t} \) (grows if \( b > d \); decays if \( b < d \))

Second Order Systems

Second order systems

\[ ay'' + by' + cy = 0 \]

assume \( a > 0 \) (otherwise multiply equation by \(-1\))

solution by Laplace transform:

\[ a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 0 \]

solve for \( Y \) (just algebraic)

\[ Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c} \]

where \( \alpha = ay(0) \) and \( \beta = ay'(0) + by(0) \)
Second Order Systems

so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1} \left( \frac{\alpha s + \beta}{\alpha s^2 + \beta s + c} \right)$$

- $\chi(s) = \alpha s^2 + \beta s + c$ is called characteristic polynomial of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial $\chi$
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

Ex: RC Circuit

Example: second-order RC circuit

at $t = 0$, the voltage across each capacitor is 1V
- for $t \geq 0$, $y$ satisfies LCCODE (from page 2-18)
  $$y'' + 3y' + y = 0$$
- initial conditions:
  $$y(0) = 1, \quad y'(0) = 0$$
  (at $t = 0$, voltage across righthand capacitor is one, current through righthand resistor is zero)
Sampling!

Not this type of sampling …

**SEMINAR REFRESHMENTS!**

Caffeine

More Caffeine

Sugar

Carbon

Sugars

Sugars with sugar

Caffeine inside the embedded in the carbs

Nothing says "We are confident this seminar will be intellectually stimulating for you" like a table full of things to help you stay awake.

JURIS G. CHAM © 2013
WWW.PHBCOMICS.COM
This type of sampling…

Analog vs Digital

- Analog Signal: An analog or analogue signal is any variable signal continuous in both time and amplitude

- Digital Signal: A digital signal is a signal that is both discrete and quantized

E.g. Music stored in a CD: 44,100 Samples per second and 16 bits to represent amplitude
Digital Signal

- Representation of a signal against a discrete set

- The set is fixed in by computing hardware

$$ s \in \mathbb{Z} $$

- Can be scaled or normalized … but is limited

$$ s \in \mathbb{Z}(0, \ldots, 2^{16}) $$

- Time is also discretized

$$ s' \in \frac{\mathbb{Z}(0, \ldots, 2^{16})}{2^{16}} $$

Representation of Signal

- Time Discretization

- Digitization
Signal: A carrier of (desired) information [1]

- Need **NOT** be electrical:
  - Thermometer
  - Clock hands
  - Automobile speedometer

- Need **NOT** always being given
  - “Abnormal” sounds/operations
  - Ex: “pitch” or “engine hum” during machining as an indicator for feeds and speeds

---

Signal: A carrier of (desired) information [2]

- Electrical signals
  - Voltage
  - Current

- **Digital signals**
  - **Convert analog electrical signals to an appropriate digital electrical message**
  - **Processing by a microcontroller or microprocessor**
Ex: Current-to-voltage conversion

- simple: Precision Resistor
  \[ i = \frac{V_{\text{measured}}}{R_{\text{known}}} \]
- better: Use an “op amp”

Mathematics of Sampling and Reconstruction

Impulse train
\[ \delta_T(t) = \sum \delta(t - n\Delta t) \]
Sampling frequency \( f_s = 1/\Delta t \)
Cut-off frequency = \( f_c \)
Mathematical Model of Sampling

- $x(t)$ multiplied by impulse train $\delta T(t)$

\[ x_c(t) = x(t)\delta_T(t) = x(t)[\delta(t) + \delta(t - \Delta t) + \delta(t - 2\Delta t) + \cdots] = \sum_n x(n\Delta t)\delta(t - n\Delta t) \]

- $x_c(t)$ is a train of impulses of height $x(t)|_{t=n\Delta t}$
Discrete Time Signal

- Image a signal...

Discrete Time Signals

- Digitization helps beat the Noise!
Discrete Time Signals

- But only so much…

Discrete Time Signals

- Can make control tricky!
Signal Manipulations

- **Shifting**
  \[ y(n) = x(n - n_0) \]

- **Reversal**
  \[ y(n) = x(-n) \]

- **Time Scaling**
  (Down Sampling)
  \[ y(M) = x(Mn) \]
  (Up Sampling)
  \[ y(n) = x\left(\frac{n}{N}\right) \]

---

Frequency Domain Analysis of Sampling

- Consider the case where the DSP performs no filtering operations
  - i.e., only passes \( xc(t) \) to the reconstruction filter
- To understand we need to look at the frequency domain
- Sampling: we know
  - multiplication in time \( \equiv \) convolution in frequency
  - \( F\{x(t)\} = X(w) \)
  - \( F\{\delta T(t)\} = \sum \delta(w - 2\pi n/\Delta t) \),
  - i.e., an impulse train in the frequency domain
Frequency Domain Analysis of Sampling

- In the frequency domain we have

\[
X_c(w) = \frac{1}{2\pi} \left( X(w) * \frac{2\pi}{\Delta t} \sum_n \delta\left(w - \frac{2\pi n}{\Delta t}\right)\right)
\]

\[
= \frac{1}{\Delta t} \sum_n X\left(w - \frac{2\pi n}{\Delta t}\right)
\]

- Let's look at an example
  - where \(X(w)\) is triangular function
  - with maximum frequency \(w_m\) rad/s
  - being sampled by an impulse train, of frequency \(w_s\) rad/s

- Fourier transform of original signal \(X(\omega)\)
  (signal spectrum)

- Fourier transform of impulse train \(\delta_T(\omega/2\pi)\) (sampling signal)

Original spectrum convolved with spectrum of impulse train
In this example it was possible to recover the original signal from the discrete-time samples.

But is this always the case?

Consider an example where the sampling frequency $w_s$ is reduced:

- i.e., $\Delta t$ is increased.
Due to overlapping replicas (aliasing), the reconstruction filter cannot recover the original spectrum. The effect of aliasing is that higher frequencies of "alias to" (appear as) lower frequencies.
Sampling Theorem

- The Nyquist criterion states:

  To prevent aliasing, a bandlimited signal of bandwidth \( w_B \) rad/s must be sampled at a rate greater than \( 2w_B \) rad/s

  \[-w_s > 2w_B\]

  Note: this is a > sign not a ≥

  Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

Time Domain Analysis of Sampling

- Frequency domain analysis of sampling is very useful to understand
  - sampling \((X(w)\sum \delta(w - 2\pi n/Δt))\)
  - reconstruction (lowpass filter removes replicas)
  - aliasing (if \(w_s ≤ 2w_B\))

- Time domain analysis can also illustrate the concepts
  - sampling a sinewave of increasing frequency
  - sampling images of a rotating wheel
A signal of the original frequency is reconstructed

A signal with a reduced frequency is recovered, i.e., the signal is aliased to a lower frequency (we recover a replica)
Sampling < Nyquist → Aliasing

Nyquist is not enough …
A little more than Nyquist is not enough …

1 Hz Sin Wave: \( \sin(2\pi t) \) \(\rightarrow\) 4 Hz Sampling

Sampled Spectrum \( w_s > 2w_m \)

Sampled Spectrum \( w_s < 2w_m \)

Original and replica spectrums overlap
Lower frequency recovered \( (w_s - w_m) \)
Temporal Aliasing

90° clockwise rotation/frame perceived

270° clockwise rotation/frame (90°) anticlockwise rotation perceived i.e., aliasing

Require LPF to ‘blur’ motion

Time Domain Analysis of Reconstruction

- Frequency domain: multiply by ideal LPF
  - ideal LPF: ‘rect’ function (gain $\Delta t$, cut off $w_c$)
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with ‘sinc’ function
  - as $F^{-1}\{\Delta t \ rect(w/w_c)\} = \Delta t w_c \ sinc(w_c t / \pi)$
  - i.e., weighted sinc on every sample
- Normally, $w_c = w_s/2$

$$x_r(t) = \sum_{n=\infty}^{\infty} x(n\Delta t)\Delta t w_c \ sinc\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$
Reconstruction

- Zero-Order Hold [ZOH]
Reconstruction

- Whittaker–Shannon interpolation formula

\[ x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \text{sinc} \left( \frac{t-nT}{T} \right) \]
Sampling and Reconstruction
Theory and Practice

- Signal is bandlimited to bandwidth WB
  - Problem: real signals are not bandlimited
    - Therefore, require (non-ideal) anti-aliasing filter
- Signal multiplied by ideal impulse train
  - problems: sample pulses have finite width
  - and not $\otimes$ in practice, but sample & hold circuit
- Samples discrete-time, continuous valued
  - Problem: require discrete values for DSP
    - Therefore, require A/D converter (quantisation)
- Ideal lowpass reconstruction (‘sinc’ interpolation)
  - problems: ideal lowpass filter not available
    - Therefore, use D/A converter and practical lowpass filter
'staircase' output from D/A converter (ZOH)

Smooth output from reconstruction filter
Example: error due to signal quantisation

Original Signal | After Anti-aliasing LPF | After Sample & Hold

After Reconstruction LPF | After D/A | After A/D

Complete practical DSP system signals
Zero Order Hold (ZOH)

- **Impulse train sampling not realisable**
  - sample pulses have finite width (say nanosecs)
- **This produces two effects,**
  - Impulse train has sinc envelope in frequency domain
  - impulse train is square wave with small duty cycle
  - Reduces amplitude of replica spectrums
    - smaller replicas to remove with reconstruction filter
- **Averaging of signal during sample time**
  - effective low pass filter of original signal
    - can reduce aliasing, but can reduce fidelity
    - negligible with most S/H
Aliasing: Another view of this

Aliasing - through sampling, two entirely different analog sinusoids take on the same “discrete time” identity

For $f[k]=\cos\Omega k$, $\Omega=\omega T$:

The period has to be less than $F_h$ (highest frequency):

Thus:

$0 \leq F \leq \frac{F_s}{2}$

$\omega_f$: aliased frequency: $\omega T = \omega_f T + 2\pi m$
Practical Anti-aliasing Filter

- Non-ideal filter
  - $w_c = w_s / 2$
- Filter usually 4th – 6th order (e.g., Butterworth)
  - so frequencies $> w_c$ may still be present
  - not higher order as phase response gets worse
- Luckily, most real signals
  - are lowpass in nature
    - signal power reduces with increasing frequency
  - e.g., speech naturally bandlimited (say $< 8$KHz)
  - Natural signals have a (approx) 1/f spectrum
  - so, in practice aliasing is not (usually) a problem

Amplitude spectrum of original signal

- Fourier transform of sampling signal (pulses have finite width)
  - $w_s = 2\pi / \Delta t$
  - $4\pi / \Delta t$
  - sinc envelope
  - Zero at harmonics
  - 1/duty cycle

- Fourier transform of sampled signal
  - $1 / \Delta t$
  - $w$

Original  Replica 1  Replica 2
Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every $\Delta t$ seconds
  2. holds that value constant until next sample
- Produces ‘staircase’ waveform, $x(n\Delta t)$

![Sample and Hold Diagram](sample_diagram.png)

Quantisation

- Analogue to digital converter (A/D)
  - Calculates nearest binary number to $x(n\Delta t)$
    - $x_q[n] = q(x(n\Delta t))$, where $q()$ is non-linear rounding fctn
  - output modeled as $x_q[n] = x(n\Delta t) + e[n]$
- Approximation process
  - therefore, loss of information (unrecoverable)
  - known as ‘quantisation noise’ ($e[n]$)
  - error reduced as number of bits in A/D increased
    - i.e., $\Delta x$, quantisation step size reduces
  - $|e[n]| \leq \frac{\Delta x}{2}$
### Input-output for 4-bit quantiser
(two’s compliment)

\[ \Delta x = \frac{2A}{2^m - 1} \]

where \( A \) = max amplitude
\( m \) = no. quantisation bits

<table>
<thead>
<tr>
<th>Digital</th>
<th>Analogue</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 0111</td>
<td></td>
</tr>
<tr>
<td>6 0110</td>
<td></td>
</tr>
<tr>
<td>5 0101</td>
<td></td>
</tr>
<tr>
<td>4 0100</td>
<td></td>
</tr>
<tr>
<td>3 0011</td>
<td></td>
</tr>
<tr>
<td>2 0010</td>
<td></td>
</tr>
<tr>
<td>1 0001</td>
<td></td>
</tr>
<tr>
<td>0 0000</td>
<td>0000</td>
</tr>
<tr>
<td>-1 1111</td>
<td>1111</td>
</tr>
<tr>
<td>-2 1110</td>
<td>1110</td>
</tr>
<tr>
<td>-3 1101</td>
<td>1101</td>
</tr>
<tr>
<td>-4 1100</td>
<td>1100</td>
</tr>
<tr>
<td>-5 1011</td>
<td>1011</td>
</tr>
<tr>
<td>-6 1010</td>
<td>1010</td>
</tr>
<tr>
<td>-7 1000</td>
<td>1000</td>
</tr>
</tbody>
</table>

\[ \Delta x \text{ quantisation step size} \]

\[ A = \text{max amplitude} \]
\[ m = \text{no. quantisation bits} \]

---

### Signal to Quantisation Noise

- To estimate SQNR we assume
  - \( e[n] \) is uncorrelated to signal and is a uniform random process
- assumptions not always correct!
  - not the only assumptions we could make…
- Also known a ‘Dynamic range’ \( (R_D) \)
  - expressed in decibels (dB)
  - ratio of power of largest signal to smallest (noise)

\[ R_D = 10 \log_{10} \left( \frac{P_{signal}}{P_{noise}} \right) \]
Dynamic Range

Need to estimate:

1. Noise power
   - uniform random process: \( P_{\text{noise}} = \Delta x^2/12 \)

2. Signal power
   - (at least) two possible assumptions
     1. sinusoidal: \( P_{\text{signal}} = A^2/2 \)
     2. zero mean Gaussian process: \( P_{\text{signal}} = \sigma^2 \)
        - Note: as \( \sigma \approx A/3 \): \( P_{\text{signal}} \approx A^2/9 \)
        - where \( \sigma^2 \) = variance, \( A \) = signal amplitude

Regardless of assumptions: \( R_D \) increases by 6dB for every bit that is added to the quantiser

1 extra bit halves \( \Delta x \)
i.e., \( 20\log_{10}(1/2) = 6\text{dB} \)

Practical Reconstruction

Two stage process:

1. Digital to analogue converter (D/A)
   - zero order hold filter
   - produces ‘staircase’ analogue output

2. Reconstruction filter
   - non-ideal filter: \( w_c = w_r/2 \)
   - further reduces replica spectrums
   - usually 4\text{th} – 6\text{th} order e.g., Butterworth
     - for acceptable phase response
D/A Converter

- Analogue output $y(t)$ is
  - convolution of output samples $y(n\Delta t)$ with $h_{ZOH}(t)$

\[
y(t) = \sum_n y(n\Delta t)h_{ZOH}(t - n\Delta t)
\]

\[
h_{ZOH}(t) = \begin{cases} 
1, & 0 \leq t < \Delta t \\
0, & \text{otherwise}
\end{cases}
\]

\[
H_{ZOH}(w) = \Delta t \exp\left(\frac{-jw\Delta t}{2}\right) \frac{\sin(w\Delta t / 2)}{w\Delta t / 2}
\]

D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional reconstruction filter required

Summary

- Theoretical model of Sampling
  - bandlimited signal ($w_B$)
  - multiplication by ideal impulse train ($w_s > 2w_B$)
    - convolution of frequency spectrums (creates replicas)
  - Ideal lowpass filter to remove replica spectrums
    - $wc = w_s / 2$
    - Sinc interpolation

- Practical systems
  - Anti-aliasing filter ($wc < w_s / 2$)
  - A/D (S/H and quantisation)
  - D/A (ZOH)
  - Reconstruction filter ($wc = w_s / 2$)

Don't confuse theory and practice!