# Signals and Vectors Systems as Maps

**ELEC 3004: Digital Linear Systems** Signals & Controls  
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Lecture 2

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Signals as Vectors

• Back to the beginning!

There is a perfect analogy between signals and vectors …

Signals are vectors!

• A vector can be represented as a sum of its components in a variety of ways, depending upon the choice of coordinate system. A signal can also be represented as a sum of its components in a variety of ways.
Signals as Vectors

- Represent them as Column Vectors

\[ x = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ \vdots \\ x[N] \end{bmatrix} \]

Signals as Vectors

- Can represent phenomena of interest in terms of signals

- Natural vector space structure (addition/substraction/norms)

- Can use norms to describe and quantify properties of signals
Signals as vectors

Signals can take real or complex values.

In both cases, a natural vector space structure:

- Can add two signals: \( x_1[n] + x_2[n] \)
- Can multiply a signal by a scalar number: \( C \cdot x[n] \)
- Form linear combinations: \( C_1 \cdot x_1[n] + C_2 \cdot x_2[n] \)

Various Types

- Audio signal (sound pressure on microphone)
- B/W video signal (light intensity on photosensor)
- Voltage/current in a circuit (measure with multimeter)
- Car speed (from tachometer)
- Robot arm position (from rotary encoder)
- Daily prices of books / air tickets / stocks
- Hourly glucose level in blood (from glucose monitor)
- Heart rate (from heart rate sensor)
Vector Refresher

\[ x \cdot y = |x||y|\cos \theta \quad (6.46) \]

- **Length:**
  \[ |x|^2 = x \cdot x \]

- **Decomposition:**
  \[ x = c_1 y + e_1 = c_2 y + e_2 \]

- **Dot Product of \( \perp \) is 0:**
  \[ x \cdot y = 0 \]

Vectors [2]

- **Magnitude and Direction**
  \[ f \cdot x = |f||x| \cos(\theta) \]

- **Component (projection) of a vector along another vector**
  \[ f = cx + e \quad \leftarrow \text{Error Vector} \]
Vectors [3]

- \( \infty \) bases given \( \mathbf{x} \)

- Which is the best one?
  \[
  f = \mathbf{c} \mathbf{x} \\
  \|\mathbf{c}\| = |f| \cos \theta \\
  \|\mathbf{c}\| \mathbf{x}^2 = |f| |\mathbf{x}| \cos \theta = f \cdot \mathbf{x} \\
  \mathbf{c} = \frac{f \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \\
  f \cdot \mathbf{x} = 0
  \]

- Can I allow more basis vectors than I have dimensions?

Vectors / Signals Can Be Multidimensional

- A signal is a quantity that varies as a function of an index set

- They can be multidimensional:
  - 1-dim, discrete index (time): \( x[n] \)
  - 1-dim, continuous index (time): \( x(t) \)
  - 2-dim, discrete (e.g., a B/W or RGB image): \( x[j, k] \)
  - 3-dim, video signal (e.g., video): \( x[j, k, n] \)
It’s Just a Linear Map

\[ u[n] \rightarrow D \rightarrow y[n] \]

- \( y[n] = 2u[n-1] \) is a linear map
- BUT \( y[n] = 2(u[n]-1) \) is NOT Why?

- Because of homogeneity!
  \[ T(au) = aT(u) \]

Linear combinations of signals
Application example: active noise cancellation

A “noise” signal, that we want to get rid of.

- At subject location, signal is
  \[ x[n] \]

- Microphone picks up signal
  \[ x_o[n] \]

- Subtract the two signals:
  \[ y(t) = x(t) - x_o(t) \]

Notice careful synchronization is needed!

---

Norms of signals

Can introduce a notion of signals being "nearby."

This is characterized by a metric (or distance function).

\[ d(x, y) \]

If compatible with the vector space structure, we have a norm.

\[ \| x - y \| \]
Examples of Norms

Can use many different norms, depending on what we want to do. The following are particularly important:

- $\ell_2$ (Euclidean) norm:
\[
\|x\|_2 = \left( \sum_{k=1}^{n} |x[k]|^2 \right)^{\frac{1}{2}} \quad \text{norm}(x, 2)
\]

- $\ell_1$ norm:
\[
\|x\|_1 = \sum_{k=1}^{n} |x[k]| \quad \text{norm}(x, 1)
\]

- $\ell_\infty$ norm:
\[
\|x\|_\infty = \max_k |x[k]| \quad \text{norm}(x, \infty)
\]

What are the differences?

Properties of norms

For any norm $\| \cdot \|$, and any signal $x$, we have:

- Linearity: if $C$ is a scalar,
\[
\|C \cdot x\| = |C| \cdot \|x\|
\]

- Subadditivity (triangle inequality):
\[
\|x + y\| \leq \|x\| + \|y\|
\]

Can use norms:

- To detect whether a signal is (approximately) zero.
- To compare two signals, and determine if they are “close.”
\[
\|x - y\| \approx 0
\]
Where are we going with this?

Signal representation by Orthogonal Signal Set

- **Orthogonal Vector Space**

⇒ A signal may be thought of as having components.
Component of a Signal

\[ f(t) \simeq cx(t) \quad t_1 \leq t \leq t_2 \]
\[ c = \frac{\int_{t_1}^{t_2} f(t)x(t) \, dt}{\int_{t_1}^{t_2} x^2(t) \, dt} = \frac{1}{E_x} \int_{t_1}^{t_2} f(t)x(t) \, dt \]

- Let’s take an example:

\[ \int_{t_1}^{t_2} f(t)x(t) \, dt = 0 \]

\[ f(t) \simeq c \sin t \quad 0 \leq t \leq 2\pi \]

\[ x(t) = \sin t \quad \text{and} \quad E_x = \int_0^{2\pi} \sin^2(t) \, dt = \pi \]

![Graph of \( f(t) \approx \frac{c}{2} \sin t \)]

Thus

\[ f(t) \approx \frac{c}{2} \sin t \quad (3.14) \]

Basis Spaces of a Signal

\[ \int_{t_1}^{t_2} x_m(t)x_n(t) \, dt = \begin{cases} 0 & m \neq n \\ \frac{2\pi}{\alpha} & m = n \end{cases} \]

\[ f(t) \simeq c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t) \]
\[ = \sum_{n=1}^{N} c_n x_n(t) \]

\[ e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t) \]
\[ c_n = \frac{\int_{t_1}^{t_2} f(t)x_n(t) \, dt}{\int_{t_1}^{t_2} x_n^2(t) \, dt} \]
\[ = \frac{1}{E_n} \int_{t_1}^{t_2} f(t)x_n(t) \, dt \quad n = 1, 2, \ldots, N \]

\[ f(t) = \sum_{n=1}^{N} c_n x_n(t) \quad t_1 \leq t \leq t_2 \]
Basis Spaces of a Signal

\[ f(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t) + \cdots \]
\[ = \sum_{n=1}^{\infty} c_nx_n(t) \quad t_1 \leq t \leq t_2 \]

- Observe that the error energy \( E_e \) generally decreases as \( N \), the number of terms, is increased because the term \( C_k^2 E_k \) is nonnegative. Hence, it is possible that the error energy \( \rightarrow 0 \) as \( N \rightarrow \infty \). When this happens, the orthogonal signal set is said to be complete.
- In this case, it’s no more an approximation but an equality.

Ex: Deblurring

- Matlab: `deconvwnr`
Then a System is a MATRIX

\[ y = Du. \]

\[
\begin{bmatrix}
  y[1] \\
  y[2] \\
  \vdots \\
  y[M]
\end{bmatrix} =
\begin{bmatrix}
  D_{11} & D_{12} & \cdots & D_{1N} \\
  D_{21} & D_{22} & \cdots & D_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  D_{M1} & D_{M2} & \cdots & D_{MN}
\end{bmatrix}
\begin{bmatrix}
  u[1] \\
  u[2] \\
  \vdots \\
  u[N]
\end{bmatrix}.
\]

\[ y[i] = \sum_j D_{ij} u[j]. \]
Recall From Last Time …
Classifications of Systems

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Causality:
Causal (physical or nonanticipative) systems

- Is one for which the output at any instant \( t_0 \) depends only on the value of the input \( x(t) \) for \( t \leq t_0 \). Ex:
  \[
  u(t) = x(t-2) \Rightarrow \text{causal} \\
  u(t) = x(t-2) + x(t+2) \Rightarrow \text{noncausal}
  \]

- A “real-time” system must be causals
  - How can it respond to future inputs?
- A prophetic system: knows future inputs and acts on it (now)
  - The output would begin before \( t_0 \)
- In some cases Noncausal maybe modelled as causal with delay
- Noncausal systems provide an upper bound on the performance of causal systems
Causality:
Looking at this from the output’s perspective…

- **Causal** = The output *before* some time $t$ does not depend on the input *after* time $t$.

Given: $y(t) = F(u(t))$

For:

$\tilde{u}(t) = u(t), \forall 0 \leq t < T$ or $[0, T)$

Then for a $T>0$:

$\rightarrow \tilde{y}(t) = y(t), \forall 0 \leq t < T$

- **A system with a memory**
  - Where past history (or derivative states) are relevant in determining the response

- **Ex:**
  - RC circuit: Dynamical
    - Clearly a function of the “capacitor’s past” (initial state) and
    - Time! (charge / discharge)
  - R circuit: is memoryless · the output of the system (recall $V=IR$) at some time $t$ only depends on the input at time $t$

- **Lumped/Distributed**
  - Lumped: Parameter is constant through the process & can be treated as a “point” in space
  - Distributed: System dimensions $\neq$ small over signal
    - Ex: waveguides, antennas, microwave tubes, etc.
Linear Time Invariant

- Linear & Time-invariant (of course - tautology!)
- Impulse response: \( h(t) = F(\delta(t)) \)
- Why?
  - Since it is linear the output response \( y(t) \) to any input \( x(t) \) is:
    \[
    x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) \, d\tau \\
    y(t) = F\left[ \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) \, d\tau \right] \Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau = x(t) * h(t)
    \]
- The output of any continuous-time LTI system is the convolution of input \( u(t) \) with the impulse response \( F(\delta(t)) \) of the system.

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Linear Dynamic [Differential] System

\[ a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m} \]

Laplace:

\[
 a_0 Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s) \\
 A(s) Y(s) = B(s) X(s)
\]

- Total response = Zero-input response + Zero-state response

\[
\begin{array}{c|c}
\text{Initial conditions} & \text{External Input} \\
\hline
\end{array}
\]
Linear Systems and ODE’s

• Linear system described by differential equation

\[
a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}
\]

• Which using Laplace Transforms can be written as

\[
a_0 Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s)
\]

\[
A(s)Y(s) = B(s)X(s)
\]

where \(A(s)\) and \(B(s)\) are polynomials in \(s\)

Unit Impulse Response

\(\delta(t)\): Impulsive excitation

\(h(t)\): characteristic mode terms

\[
\delta(t) \rightarrow \text{LTI} \rightarrow F(\delta(t)) \rightarrow h(t) = F(\delta(t))
\]

Ex:

\[
(30 + 30 + 2) y_0(t) = 4 x_0(t)
\]

This is a second-order system \((n = 2)\) having the characteristic polynomial \((s^2 + 3s + 2)\) being \(s = -1\) and \(s = -2\). Therefore

\[
y(0) = c_1 e^{-t} + c_2 e^{-2t}
\]

Differentiation of the equation yields

\[
y_1(t) = -c_1 e^{-t} - 2c_2 e^{-2t}
\]

The initial conditions are listed in Eqn. (2.20) for \(n = 2\)

\[
y(0) = c_1 = 1 \quad \text{and} \quad y_1(0) = 0
\]

Solving \# 2 with Eqs. (2.20a) and (2.20b) and substituting the initial conditions just given, we obtain

\[
c_1 = 1 \quad \text{and} \quad c_2 = -1
\]

\[
y(0) = e^{-t} - e^{-2t}
\]

Moreover, according to Eqn. (2.22), \(F(\delta(t)) = \Delta(t)\) so that

\[
F(F(\delta(t))) = F(\delta(t)) = e^{-t} - e^{-2t}
\]

Also, in this case, \(s = 0\) (zero second-order term is absent in PDE). Therefore

\[
\delta(t) = F(F(\delta(t))) = e^{-t} - e^{-2t}
\]
System Models

- Various things – all the same!

### Table 2.1 Summary of Through- and Across-Variables for Physical Systems

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<th>Variable Across Element</th>
<th>Integrated Across-Variable</th>
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<td>Volume, $V$</td>
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<td>Heat energy, $H$</td>
<td>Temperature difference, $\delta_{21}$</td>
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### Circuits

\[
\frac{v_{out}}{v_{in}} = \frac{1}{C_1C_2 R_1 R_2 s^2 + C_2 (R_1 + R_2) s + 1}
\]

\[
\frac{V_2(s)}{V_1(s)} \approx \frac{1}{RCs}
\]
**Motors**

5. DC motor, field-controlled, rotational actuator

\[
\theta(s) = \frac{K_m}{V_f(s)} = \frac{K_m}{s(Js + b)(L_s + R_f)}
\]

7. AC motor, two-phase control field, rotational actuator

\[
\theta(s) = \frac{K_m}{V_f(s)} = \frac{K_m}{s(\tau s + 1)}
\]

\[
\tau = J/(b - m)
\]

\[
m = \text{slope of linearized torque-speed curve (normally negative)}
\]

**Mechanical Systems**

15. Accelerometer, acceleration sensor

\[
x_a(t) = y(t) - x_a(t)
\]

\[
X_a(s) = \frac{-s^2}{s^2 + (b/M)s + k/M}
\]

For low-frequency oscillations, where \( \omega < \omega_a \),

\[
X_a(j\omega) \approx \frac{-\omega^2}{k/M}
\]
Thermal Systems

16. Thermal heating system

\[ \frac{\overline{q}(s)}{s} = \frac{1}{C_T + (QS + 1/R_t)}, \]

where

- \( C_T \) = thermal capacitance
- \( Q \) = fluid flow rate = constant
- \( S \) = specific heat of water
- \( R_t \) = thermal resistance of insulation

\( q(s) \) = transform of rate of heat flow of heating element

First Order Systems

First order systems

\[ ay' + by = 0 \quad (\text{with } a \neq 0) \]

righthand side is zero:

- called autonomous system
- solution is called natural or unforced response

can be expressed as

\[ Ty' + y = 0 \quad \text{or} \quad y' + ry = 0 \]

where

- \( T = a/b \) is a time (units: seconds)
- \( r = b/a = 1/T \) is a rate (units: 1/sec)
First Order Systems

Solution by Laplace transform

take Laplace transform of \( Ty' + y = 0 \) to get

\[
T(sY(s) - y(0)) + Y(s) = 0
\]

solve for \( Y(s) \) (algebraic)

\[
Y(s) = \frac{Ty(0)}{sT + 1} = \frac{y(0)}{s + 1/T}
\]

and so \( y(t) = y(0)e^{-t/T} \)

---

First Order Systems

solution of \( Ty' + y = 0 \): \( y(t) = y(0)e^{-t/T} \)

if \( T > 0 \), \( y \) decays exponentially

- \( T \) gives time to decay by \( e^{-1} \approx 0.37 \)
- \( 0.693T \) gives time to decay by half (\( 0.693 = \log 2 \))
- \( 4.6T \) gives time to decay by 0.01 (\( 4.6 = \log 100 \))

if \( T < 0 \), \( y \) grows exponentially

- \( |T| \) gives time to grow by \( e \approx 2.72 \)
- \( 0.693|T| \) gives time to double
- \( 4.6|T| \) gives time to grow by 100
First Order Systems

Examples

simple RC circuit:

\[
\begin{array}{c}
R \\
C \\
\hline
+ \\
\hline
v \\
- \\
\end{array}
\]

circuit equation: \( RCv' + v = 0 \)

solution: \( v(t) = v(0)e^{-t/(RC)} \)

population dynamics:

- \( y(t) \) is population of some bacteria at time \( t \)
- growth (or decay if negative) rate is \( y' = by - dy \) where \( b \) is birth rate, \( d \) is death rate
- \( y(t) = y(0)e^{(b-d)t} \) (grows if \( b > d \); decays if \( b < d \))

Second Order Systems

Second order systems

\[ ay'' + by' + cy = 0 \]

assume \( a > 0 \) (otherwise multiply equation by \(-1\))

solution by Laplace transform:

\[
a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 0
\]

solve for \( Y \) (just algebraic)

\[
Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}
\]

where \( \alpha = ay(0) \) and \( \beta = ay'(0) + by(0) \)
Second Order Systems

so solution of \( ay'' + by' + cy = 0 \) is

\[
y(t) = \mathcal{L}^{-1} \left( \frac{\alpha s + \beta}{s^2 + bs + c} \right)
\]

- \( \chi(s) = as^2 + bs + c \) is called characteristic polynomial of the system
- form of \( y = \mathcal{L}^{-1}(Y) \) depends on roots of characteristic polynomial \( \chi \)
- coefficients of numerator \( \alpha s + \beta \) come from initial conditions

Ex: RC Circuit

Example: second-order RC circuit

\[
\begin{align*}
\text{at } t = 0, \text{ the voltage across each capacitor is } 1V \\
\text{for } t \geq 0, y \text{ satisfies LCCODE (from page 2-18)} \\
\quad g'' + 3g' + y = 0 \\
\text{initial conditions: } g(0) = 1, \quad g'(0) = 0
\end{align*}
\]

(at \( t = 0 \), voltage across righthand capacitor is one; current through righthand resistor is zero)
Next Time…

• Sampling
  – Measurements at regular intervals of a continuous signal
  – Not to be confused with
    “How to try regional dishes without indigestion”

• Review:
  – Chapter 8 of Lathi

• Send (and you shall receive) a positive signal 😊