**PID Control & Estimation**

ELEC 3004: **Digital Linear Systems** Signals & Controls
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Lecture 12

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May 27, 2014
Announcements:

• **PS 3 Grading:**
  - Due Wednesday at 11:59pm
  - We’re working on it

• **PS 4:**
  - Working on it too!

• **Lab 4:**
  - Working on it too too!!

• **Final Exam Logistics:**
  - Saturday 21/6/2014 at 4:30pm
  - Location: TBA
  - Closed-book
  - Practice exam will be posted soon
  (Working on it too too too!!)
PID

- Three basic types of control:
  - Proportional
  - Integral, and
  - Derivative

- The next step up from lead compensation
  - Essentially a combination of proportional and derivative control

![PID Diagram](image)

Proportional Control

A discrete implementation of proportional control is identical to continuous; that is, where the continuous is

\[ u(t) = K_p e(t) \Rightarrow D(s) = K_p, \]

the discrete is

\[ u(k) = K_p e(k) \Rightarrow D(z) = K_p \]

where \( e(t) \) is the error signal as shown in Fig 5.2.

![Proportional Control Diagram](image)
### Derivative Control

For continuous systems, derivative or rate control has the form

\[ u(t) = K_p T_D \dot{e}(t) \Rightarrow D(s) = K_p T_D s \]

where \( T_D \) is called the derivative time. Differentiation can be approximated in the discrete domain as the first difference, that is,

\[ u(k) = K_p T_D \frac{e(k) - e(k-1)}{T} \Rightarrow D(z) = K_p T_D \frac{1 - z^{-1}}{T} = K_p T_D \frac{z - 1}{T z}. \]

In many designs, the compensation is a sum of proportional and derivative control (or PD control). In this case, we have

\[ D(z) = K_p \left( 1 + \frac{T_D(z - 1)}{T z} \right). \]

or, equivalently,

\[ D(z) = K_p \frac{z - \alpha}{z} \]

### Derivative Control [2]

- **Similar to the lead compensators**
  - The difference is that the pole is at \( z = 0 \)

  [ Whereas the pole has been placed at various locations along the \( z \)-plane real axis for the previous designs. ]

- **In the continuous case:**
  - pure derivative control represents the ideal situation in that there is no destabilizing phase lag from the differentiation
  - the pole is at \( s = -\infty \)

- **In the discrete case:**
  - \( z=0 \)
  - However this has phase lag because of the necessity to wait for one cycle in order to compute the first difference
Integral Control

For continuous systems, we integrate the error to arrive at the control,

\[ u(t) = \frac{K_p}{T_I} \int_{t_0}^{t} e(t) dt \Rightarrow D(s) = \frac{K_p}{T_I s}, \]

where \( T_I \) is called the integral, or reset time. The discrete equivalent is to sum all previous errors, yielding

\[ u(k) = u(k-1) + \frac{K_p T_I}{T_I} e(k) \Rightarrow D(z) = \frac{K_p T}{T_I (1 - z^{-1})} = \frac{K_p T z}{T_I (z - 1)}. \] (5.60)

Just as for continuous systems, the primary reason for integral control is to reduce or eliminate steady-state errors, but this typically occurs at the cost of reduced stability.

PID Control

\[ D(z) = K_p \left( 1 + \frac{T_z}{T_I (z - 1)} + \frac{T_D (z - 1)}{T_I z} \right). \]

The user simply has to determine the best values of

- \( K_p \)
- \( T_D \) and
- \( T_I \)
PID as Difference Equation

\[
\frac{U(z)}{E(z)} = D(z) = K_p + K_i \left( \frac{T_z}{z - 1} \right) + K_d \left( \frac{z - 1}{T_z} \right)
\]

\[
u(k) = \left[ K_p + K_i T + \left( \frac{K_d}{T} \right) \right] \cdot e(k) - [K_d T] \cdot e(k - 1) + [K_i] \cdot u(k - 1)
\]

State-Space Control

\[
\dot{x} = Fx
\]

(That can not be all of it? There has to be more to it than this...)
State-Space Control

\[
\dot{x} = Fx + Gu
\]

**Benefits:**
- Characterises the process by systems of coupled, first-order differential equations
- More general mathematical model
  - MIMO, time-varying, nonlinear
- Mathematically esoteric (who needs practical solutions)
- Yet, well suited for digital computer implementation
  - That is: based on vectors/matrices (think LAPACK → MATLAB)

---

**Difference Equations & Feedback**

- Start with the Open-Loop:
  \[ y = Hu \]
- Close the loop:
  \[
  u = ke = k(\hat{y} - y) \Rightarrow y = H[k(\hat{y} - y)]
  \]
  \[ \Rightarrow y = \frac{Hk}{1+Hk} \hat{y} \]
- All easy! (yesa!)
Difference Equations & Feedback

- Now add delay (image the plant is a replica with delay $\tau$)
  \[ y(t) = u(t - \tau) \]

- Close the loop:
  \[ u(t - \tau) = ke(t - \tau) = k [\hat{y}(t - \tau) - y(t - \tau)] \]
  \[ \Rightarrow y(t) = k [\hat{y}(t - \tau) - y(t - \tau)] \]

- Notice we have a difference equation!

- What happens with a single delay and a unit step?
  \[ u(t) = k \text{ for } 0 < t < \tau \]
  \[ y(t) = u(t - \tau) \text{ for } \tau < t < 2\tau \]

- Then with feedback we get:
  \[ u(t) = k(1 - k) = k - k^2 \]
  \[ y(t) = k - k^2 + k^3 + \cdots + (-1)^{n-1}k^{n-1} \]

- If $k < 1$: then:
  \[ \lim_{t \to \infty} y(t) = \frac{k}{1+k} \]
Great, so how about control?

- Given $\dot{x} = Fx + Gu$, if we know $F$ and $G$, we can design a controller $u = -Kx$ such that
  $$\text{eig}(F - GK) < 0$$

- In fact, if we have full measurement and control of the states of $x$, we can position the poles of the system in arbitrary locations!
Example: PID control

- Consider a system parameterised by three states:
  - $x_1, x_2, x_3$
  - where $x_2 = \dot{x}_1$ and $x_3 = \dot{x}_2$

$$\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 \end{bmatrix} x - Ku$$

$$y = [0 \ 1 \ 0]x + 0u$$

$x_2$ is the output state of the system;
$x_1$ is the value of the integral;
$x_3$ is the velocity.

- We can choose $K$ to move the eigenvalues of the system as desired:

$$\det \begin{bmatrix} 1 - K_1 & 1 - K_2 \\ 1 - K_2 & -2 - K_3 \end{bmatrix} = 0$$

All of these eigenvalues must be positive.

It’s straightforward to see how adding derivative gain $K_3$ can stabilise the system.
Example 1: Command Shaping

Command Shaping

Position

Time

A1
A2
0.1
0.2
0.3
0.4
0.5
0.6
-0.1
-0.2
-0.3
-0.4
0
0.5
1
1.5
2
2.5
3

A1 Response
A2 Response
Total Response
Command Shaping

- Zero Vibration (ZV)
  \[ A_i \begin{bmatrix} 1 \\ 1+K \\ 0 \end{bmatrix} + \begin{bmatrix} K \\ 1+K \\ T_d/2 \end{bmatrix} = K e^{-\zeta \pi \sqrt{1-\zeta^2}} \]

- Zero Vibration and Derivative (ZVD)
  \[ A_i \begin{bmatrix} (1+K)^2 \\ 0 \\ (1+K)^2 \end{bmatrix} + \begin{bmatrix} 2K \\ T_d/2 \\ T_d \end{bmatrix} \]

Can you use this for more than Control?

- Yes
The Approach:

- Formulate the goal of control as an **optimization** (e.g. minimal impulse response, minimal effort, ...).
- You’ve already seen some examples of optimization-based design:
  - Used least-squares to obtain an FIR system which matched (in the least-squares sense) the desired frequency response.
  - Poles/zeros lecture: Butterworth filter
“Maximally-flat filter”. Sacrifice sharpness to have flat response in pass band and stop band.

![Magnitude vs Frequency](image1.png)

![Magnitude vs Imaginary](image2.png)

![Magnitude vs Real](image3.png)
“Maximally-flat filter”. Sacrifice sharpness to have flat response in pass band and stop band.
“Maximally-flat filter”. Sacrifice sharpness to have flat response in pass band and stop band.

![Graph showing frequency response and magnitude](image1)

![Diagram showing complex plane](image2)
How?

- Constrained Least-Squares ...

  One formulation: Given \( x[0] \)

  \[
  \min_{u[0], u[1], \ldots, u[N]} ||\overline{u}||^2, \quad \text{where} \quad \overline{u} = \begin{bmatrix}
  u[0] \\
  u[1] \\
  \vdots \\
  u[N]
  \end{bmatrix}
  \]

  subject to \( x[N] = 0. \)

  Note that

  \[
  x[n] = A^n x[0] + \sum_{k=0}^{n-1} A^{(n-1-k)} B u[k],
  \]

  so this problem can be written as

  \[
  \min_{x_{ls}} ||A_{ls} x_{ls} - b_{ls}||^2 \quad \text{subject to} \quad C_{ls} x_{ls} = D_{ls}.
  \]

---

Shannon Information Theory

On the transmission of information over a noisy channel:

- An information source that produces a message
- A transmitter that operates on the message to create a signal which can be sent through a channel
- A channel, which is the medium over which the signal, carrying the information that composes the message, is sent
- A receiver, which transforms the signal back into the message intended for delivery
- A destination, which can be a person or a machine, for whom or which the message is intended
Additional Use II: Estimation

Along multiple dimensions
State Space

- We collect our set of uncertain variables into a vector ...
  \[ x = [x_1, x_2, \ldots, x_N]^T \]

- The set of values that \( x \) might take on is termed the \textit{state space}

- There is a \textit{single} true value for \( x \), but it is unknown

State Space Dynamics

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
H(s) &= C(sI - A)^{-1}B
\end{align*}
\]
Measured versus True

- Measurement errors are inevitable

- So, add Noise to State...
  - State Dynamics becomes:

\[
\begin{align*}
\dot{x} &= Ax + Bu + w \\
y &= Cx + Du + v
\end{align*}
\]

- Can represent this as a normal Distribution

\[
N(x; \mu, \sigma) = \frac{1}{(\sqrt{2\pi})\sigma} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]

Recovering The Truth

- Numerous methods
- Termed “Estimation” because we are trying to estimate the truth from the signal

- A strategy discovered by Gauss
- Least Squares in Matrix Representation

\[
\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \left[ \sum_{i=1}^{n} t_i \sum_{i=1}^{n} t_i^2 \right]^{-1} \left[ \sum_{i=1}^{n} z_i t_i \sum_{i=1}^{n} t_i z_i \right]
\]
Recovering the Truth: Terminology

\[ \dot{x} = Fx + Gu + w \]
\[ z = Hx + v \]

- \( x \) : the state vector
- \( x_{k+1} \) : the state of \( x \) at time \( A \) based on data taken up to time \( B \)
- \( \hat{x} \) : estimate of the true state vector
- \( F \) : system dynamics matrix in continuous time (equivalent to \( A \) in Eq. 1)
- \( G \) : system control matrix relating deterministic input, \( u \), to the state (equivalent to \( B \) in Eq. 1)
- \( H \) : measurement matrix in continuous time (equivalent to \( C \) in Eq. 2)
- \( F_t \) : system model in discrete time at \( t = t_i \)
- \( P_t \) : estimate covariance in discrete time at \( t = t_i \)
- \( w \) : process uncertainty (noise) vector (of type \( N(0, Q) \))
- \( Q \) : process noise matrix, \( Q = \begin{bmatrix} \sigma^2 \end{bmatrix} \)
- \( Q_t \) : \( Q \) in discrete time at \( t = t_i \)
- \( v \) : measurement noise vectors (of type \( N(0, R) \))
- \( H_t \) : the measurement variance matrix, \( R = \begin{bmatrix} \nu \end{bmatrix} \)

General Problem...

True state

\[ x_{k+1} \]
\[ F_{k+1} \]
\[ x_{k+1} \]
\[ F \]
\[ x_k \]
\[ H \]
\[ z_k \]

Observe state

\[ z_{k+1} \]
\[ z_{k+1} \]
\[ z_k \]

\( t \)
## Duals and Dual Terminology

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<td>$\dot{x} = Fx$ (discrete: $\dot{x} = F_k x$)</td>
<td>$\dot{x} = Ax$, $A = F^1$</td>
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<td>Regulates:</td>
<td>$P$ (covariance)</td>
<td>$M$ (performance matrix)</td>
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<td>Minimized function:</td>
<td>$Q$ (or $GGQ^T$)</td>
<td>$V$</td>
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<td>Optimal Gain:</td>
<td>$K$</td>
<td>$G$</td>
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<td>Completeness law:</td>
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### Estimation Process in Pictures

**System:** (unknown)

- $X_{k-1}$
- $X_k$

**Measured:**

- $v \sim R_k \equiv N(0,r)$
- $w \sim Q_k$
- $Z_k = U$

**Estimate:**

- $\hat{X}_{k-1}$
- $\hat{X}_k$
- $P_{k-1}$
- $P_k$
- $F_k + W$
Kalman Filter Process

KF Process in Equations

Prediction: $\hat{x}_{k|k-1} = F_{k-1} x_{k-1|k-1}$,  
$P_{k|k-1} = Q_{k-1} + F_{k-1} P_{k-1|k-1} F_{k-1}^T$,  
Kalman Gain: $K_k = P_{k|k-1} H^T [HP_{k|k-1} H^T + R_k]^{-1}$,  
Update: $P_{k|k} = [I - K_k H] P_{k|k-1}$,  
$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (z_k - H \hat{x}_{k|k-1})$
KF Considerations

\[ \dot{x}_{k|k-1} = F_{k|k-1}x_{k-1|k-1} + G_{k-1}u_{k-1} \]
\[ P_{k|k-1} = Q_{k-1} + F_{k|k-1}P_{k-1|k-1}F_{k|k-1}^T \]
\[ K_k = P_{k|k-1}H^T[HP_{k|k-1}H^T + R_k]^{-1} \]
\[ P_{k|k} = [I - K_kH]P_{k|k-1} \]
\[ \dot{x}_{k|k} = \dot{x}_{k|k-1} + K_k(z_k - H\dot{x}_{k|k-1} -HG_ku_{k-1}) \]

Ex: Kinematic KF: Tracking

- Consider a System with Constant Acceleration

\[ \ddot{y} = -g \]
\[ \dot{y} = g t + p_1 \]
\[ y = p_0 + p_1 t + \frac{g t^2}{2} \]

\[ \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} g \\ \end{bmatrix} \]

\[ F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ F_k = \begin{bmatrix} 0 & t_s & \frac{t_s^2}{2} \\ 0 & 0 & t_s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ \dot{x}_k = F_{k-1}\dot{x}_{k-1} + K_k(x_k - HF_{k-1}\dot{x}_{k-1}) \]
In Summary

• KF:
  – The true state \((x)\) is separate from the measured \((z)\)
  – Lets you **combine** prior controls knowledge with measurements to filter signals and find the **truth**
  – It **regulates** the covariance \((P)\)
    • As \(P\) is the scatter between \(z\) and \(x\)
    • So, if \(P \rightarrow 0\), then \(z \rightarrow x\) (measurements \(\rightarrow\) truth)

• EKF:
  – Takes a Taylor series approximation to get a local “F” (and “G” and “H”)