Sampling & Data Acquisition

ELEC 3004: Systems: Signals & Controls
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Lecture 6

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Announcements:

• Assignment 1 is Posted:
  – While some questions are “shocking”
  – Most are designed to be fair – not trying to be tricky

⇒ For next Lecture (#7):
  – Review Fourier Transform materials
  – Please review Convolution

• ELEC 7213 Students:
  – You have 2 additional questions
  – Submission instructions for Q9-Q10 via email.

• Space Audit on March 22 (F) and March 28 (W)
  – Sounds like an ideal time for a pop-quiz
  – Bring a friend or two ( just saying 😊 )
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Analog vs Digital

- **Analog Signal**: An analog or analogue signal is any variable signal continuous in both time and amplitude

- **Digital Signal**: A digital signal is a signal that is both discrete and quantized

E.g. Music stored in a CD: 44,100 Samples per second and 16 bits to represent amplitude
Digital Signal

- Representation of a signal against a discrete set

- The set is fixed in by computing hardware

\[ S \in \mathbb{Z} \]

- Can be scaled or normalized ... but is limited

\[ s \in \mathbb{Z}(0, \ldots, 2^{16}) \]

- Time is also discretized

\[ s' \in \mathbb{Z}(0, \ldots, 2^{16})/2^{16} \]

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Representation of Signal

- Time Discretization

- Digitization
**Signal: A carrier of (desired) information [1]**

- **Need NOT be electrical:**
  - Thermometer
  - Clock hands
  - Automobile speedometer

- **Need NOT always being given**
  - “Abnormal” sounds/operations
  - Ex: “pitch” or “engine hum” during machining as an indicator for feeds and speeds

**Signal: A carrier of (desired) information [2]**

- **Electrical signals**
  - Voltage
  - Current

- **Digital signals**
  - Convert analog electrical signals to an appropriate digital electrical message
  - Processing by a microcontroller or microprocessor
Transduction (sensor to an electrical signal)

- Sensor reacts to environment (physics)
- Turn this into an electrical signal:
  - V: voltage source
  - I: current source
- Measure this signal
  - Resistance
  - Capacitance
  - Inductance

Ex: Current-to-voltage conversion

- simple:
  Precision Resistor
- better:
  Use an “op amp”

\[ i = \frac{V_{\text{measured}}}{R_{\text{known}}} \]
Mathematics of Sampling and Reconstruction

[Diagram showing sampling and reconstruction]

Impulse train
\[ \delta_T(t) = \sum \delta(t - n\Delta t) \]

Sampling frequency \( f_s = 1/\Delta t \)

Cut-off frequency = \( f_c \)

Mathematical Model of Sampling

- \( x(t) \) multiplied by impulse train \( \delta_T(t) \)

\[
x_c(t) = x(t)\delta_T(t) = x(t)\left[ \delta(t) + \delta(t - \Delta t) + \delta(t - 2\Delta t) + \cdots \right]
\]

\[
= \sum_n x(n\Delta t)\delta(t - n\Delta t)
\]

- \( x_c(t) \) is a train of impulses of height \( x(t)|_{t=n\Delta t} \)
Consider the case where the DSP performs no filtering operations — i.e., only passes $x_c(t)$ to the reconstruction filter.

To understand we need to look at the frequency domain.

Sampling: we know
- multiplication in time $\Leftrightarrow$ convolution in frequency
- $F\{x(t)\} = X(w)$
- $F\{\delta_T(t)\} = \sum \delta(w - 2\pi n/\Delta t)$,
  - i.e., an impulse train in the frequency domain.

Frequency Domain Analysis of Sampling
**Frequency Domain Analysis of Sampling**

- In the frequency domain we have

\[ X_c(w) = \frac{1}{2\pi} \left( X(w) * \frac{2\pi}{\Delta t} \sum_n \delta \left( w - \frac{2\pi n}{\Delta t} \right) \right) \]

\[ = \frac{1}{\Delta t} \sum_n X \left( w - \frac{2\pi n}{\Delta t} \right) \]

- Let's look at an example
  - where \( X(w) \) is triangular function
  - with maximum frequency \( w_m \text{ rad/s} \)
  - being sampled by an impulse train, of frequency \( w_s \text{ rad/s} \)

- Remember convolution with an impulse?
- Same idea for an impulse train

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**Fourier transform of original signal** \( X(\omega) \) (signal spectrum)

- \( X(\omega) \)
- \(-w_m\)
- \( w_m\)
- \( w\)

**Fourier transform of impulse train** \( \delta_T(\omega/2\pi) \) (sampling signal)

- \( \delta_T(\omega/2\pi) \)
- \( 0\)
- \( w_c = 2\pi/\Delta t\)
- \( 4\pi/\Delta t\)

**Fourier transform of sampled signal**

- Original spectrum convolved with spectrum of impulse train

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Original | Replica 1 | Replica 2
--- | --- | ---

**Original**

\[ \frac{1}{\Delta t} \]

**Replica 1**

**Replica 2**
Spectrum of sampled signal

Original Replica 1 Replica 2

Reconstruction filter (ideal lowpass filter)

Reconstruction filter removes the replica spectrums & leaves only the original

\[ X(\omega) = H_L(\omega) X_c(\omega) \]

Sampling Frequency

- In this example it was possible to recover the original signal from the discrete-time samples
- But is this always the case?
- Consider an example where the sampling frequency \( w_s \) is reduced
  - i.e., \( \Delta t \) is increased
Due to overlapping replicas (aliasing), the reconstruction filter cannot recover the original spectrum. The effect of aliasing is that higher frequencies of “alias to” (appear as) lower frequencies.
The Nyquist criterion states:

To prevent aliasing, a **bandlimited** signal of bandwidth $w_B$ rad/s must be sampled at a rate greater than $2w_B$ rad/s

$$-w_s > 2w_B$$

**Note:** this is a $>$ sign not a $\geq$

Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

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**Time Domain Analysis of Sampling**

- Frequency domain analysis of sampling is very useful to understand
  - sampling ($X(w)\sum \delta(w - \frac{2\pi n}{\Delta t})$)
  - reconstruction (lowpass filter removes replicas)
  - aliasing (if $w_s \leq 2w_B$)
- Time domain analysis can also illustrate the concepts
  - sampling a sinewave of increasing frequency
  - sampling images of a rotating wheel
A signal of the original frequency is reconstructed

A signal with a reduced frequency is recovered, i.e., the signal is aliased to a lower frequency (we recover a replica)
Sampling < Nyquist $\rightarrow$ Aliasing

Nyquist is not enough …
A little more than Nyquist is not enough ...

1 Hz Sin Wave: $\sin(2\pi t) \rightarrow 4$ Hz Sampling

Sampled Spectrum $w_s > 2w_m$

LPF

original

$-w_m$ $w_m$ $w_s$

replica 1

original freq recovered

Sampled Spectrum $w_s < 2w_m$

LPF

original

$-w_m$ $w_m$ $w_s$

replica 1

Original and replica spectrums overlap
Lower frequency recovered ($w_s - w_m$)
Temporal Aliasing

90° clockwise rotation/frame clockwise rotation perceived

270° clockwise rotation/frame (90°) anticlockwise rotation perceived i.e., aliasing

Require LPF to ‘blur’ motion

Time Domain Analysis of Reconstruction

- Frequency domain: multiply by ideal LPF
  - ideal LPF: ‘rect’ function (gain Δt, cut off \( w_c \))
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with ‘sinc’ function
  - as \( F^{-1}\{Δt \text{ rect}(w/w_c)\} = Δt w_c \text{ sinc}(w_c t/\pi) \)
  - i.e., weighted sinc on every sample
- Normally, \( w_c = w_s/2 \)

\[
x_r(t) = \sum_{n=\infty}^{\infty} x(n\Delta t)\Delta t w_c \text{sinc} \left( \frac{w_c(t-n\Delta t)}{\pi} \right)
\]
Sampling and Reconstruction
Theory and Practice

- Signal is bandlimited to bandwidth $W_B$
  - Problem: real signals are not bandlimited
    - Therefore, require (non-ideal) anti-aliasing filter
- Signal multiplied by ideal impulse train
  - Problems: sample pulses have finite width
  - And not $\otimes$ in practice, but sample & hold circuit
- Samples discrete-time, continuous valued
  - Problem: require discrete values for DSP
    - Therefore, require A/D converter (quantisation)
- Ideal lowpass reconstruction (‘sinc’ interpolation)
  - Problems: ideal lowpass filter not available
    - Therefore, use D/A converter and practical lowpass filter
Practical DSP System

Note: $w_s > 2w_B$

Practical Anti-aliasing Filter

- Non-ideal filter
  - $w_c = w_s / 2$
- Filter usually 4th – 6th order (e.g., Butterworth)
  - so frequencies $> w_c$ may still be present
  - not higher order as phase response gets worse
- Luckily, most real signals
  - are lowpass in nature
    - signal power reduces with increasing frequency
    - e.g., speech naturally bandlimited (say < 8KHz)
  - Natural signals have a (approx) 1/f spectrum
  - so, in practice aliasing is not (usually) a problem
Finite Width Sampling

- Impulse train sampling not realisable
  - sample pulses have finite width (say nanosecs)
- This produces two effects,
- Impulse train has sinc envelope in frequency domain
  - impulse train is square wave with small duty cycle
  - Reduces amplitude of replica spectrums
    - smaller replicas to remove with reconstruction filter 😊
- Averaging of signal during sample time
  - effective low pass filter of original signal
    - can reduce aliasing, but can reduce fidelity 😞
    - negligible with most S/H 😊
Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every $\Delta t$ seconds
  2. holds that value constant until next sample
- Produces ‘staircase’ waveform, $x(n\Delta t)$

Quantisation

- Analogue to digital converter (A/D)
  - Calculates nearest binary number to $x(n\Delta t)$
    - $x[n] = q(x(n\Delta t))$, where $q()$ is non-linear rounding fctn
    - output modeled as $x[n] = x(n\Delta t) + e[n]$
  - Approximation process
    - therefore, loss of information (unrecoverable)
    - known as ‘quantisation noise’ ($e[n]$)
    - error reduced as number of bits in A/D increased
      - i.e., $\Delta x$, quantisation step size reduces

$$|e[n]| \leq \frac{\Delta x}{2}$$
## Input-output for 4-bit quantiser (two’s compliment)

\[ \Delta x = \frac{2A}{2^m - 1} \]

where \( A \) = max amplitude  
\( m \) = no. quantisation bits

\[ \Delta x \] quantisation step size

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<td>-7</td>
<td>1000</td>
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### Example: error due to signal quantisation

- original signal \( x(t) \)
- quantised samples \( x_q(t) \)

**Figure:**
- Sample number
- Amplitude (V)
Signal to Quantisation Noise

- To estimate SQNR we assume:
  - $e[n]$ is uncorrelated to signal and is a uniform random process
- assumptions not always correct!
  - not the only assumptions we could make…

- Also known as ‘Dynamic range’ ($R_D$)
  - expressed in decibels (dB)
  - ratio of power of largest signal to smallest (noise)

\[
R_D = 10 \log_{10} \left( \frac{P_{signal}}{P_{noise}} \right)
\]

Dynamic Range

Need to estimate:

1. Noise power
   - uniform random process: $P_{noise} = \Delta x^2/12$
2. Signal power
   - (at least) two possible assumptions
     1. sinusoidal: $P_{signal} = A^2/2$
     2. zero mean Gaussian process: $P_{signal} = \sigma^2$
       • Note: as $\sigma = A/3$: $P_{signal} = A^2/9$
       • where $\sigma^2 = \text{variance, } A = \text{signal amplitude}$

Regardless of assumptions: $R_D$ increases by 6dB for every bit that is added to the quantiser
Practical Reconstruction

Two stage process:
1. Digital to analogue converter (D/A)
   - zero order hold filter
   - produces ‘staircase’ analogue output
2. Reconstruction filter
   - non-ideal filter: \( w_c = w_s / 2 \)
   - further reduces replica spectrums
   - usually 4th – 6th order e.g., Butterworth
     - for acceptable phase response

D/A Converter

- Analogue output \( y(t) \) is
  - convolution of output samples \( y(n\Delta t) \) with \( h_{ZOH}(t) \)

\[
y(t) = \sum_n y(n\Delta t)h_{ZOH}(t - n\Delta t)
\]

\[
h_{ZOH}(t) = \begin{cases} 
1, & 0 \leq t < \Delta t \\
0, & \text{otherwise}
\end{cases}
\]

\[
H_{ZOH}(w) = \Delta t \exp \left( -jw\Delta t \right) \frac{\sin(w\Delta t / 2)}{w\Delta t / 2}
\]

D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional reconstruction filter required
Zero Order Hold (ZOH)

ZOH impulse response

ZOH amplitude response

ZOH phase response

'staircase' output from D/A converter (ZOH)
Smooth output from reconstruction filter

![Graph showing smooth output from reconstruction filter](image)

**Graph Details:**
- **X-axis:** Time (sec)
- **Y-axis:** Amplitude (V)
- **Legend:**
  - D/A output
  - Reconstruction filter output

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Complete practical DSP system signals

![Diagram showing complete practical DSP system signals](image)
Summary

- Theoretical model of Sampling
  - bandlimited signal (wB)
  - multiplication by ideal impulse train (ws > 2wB)
    - convolution of frequency spectrums (creates replicas)
  - Ideal lowpass filter to remove replica spectrums
    - wc = ws / 2
    - Sinc interpolation

- Practical systems
  - Anti-aliasing filter (wc < ws / 2)
  - A/D (S/H and quantisation)
  - D/A (ZOH)
  - Reconstruction filter (wc = ws / 2)

Don't confuse theory and practice!
Next Time…

• Back to Systems Modelling
  – Convolution

• Review:
  – Latter Parts of Chapter 2 of Lathi

• Send (and you shall receive) a positive signal 😊