Noise & Filters

ELEC 3004: Systems: Signals & Controls
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Lecture 12

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Goals for the Week

• Continue of Discussion of the z-Transform

• Noise

• Introduce Filters (Today) ➔ Analog Filters Friday
  – Frequency Response of an LTIC and LTID System
  – Butterworth
  – Chebyshev

• Digital Filters IIR Filters
  • Design of IIR Filters from Analog Filters
    – FIR Filter
The z-Transform

- It is defined by:
  \[ z = r e^{j\omega} \]

  Or in the Laplace domain:
  \[ z = e^{sT} \]

- Thus:
  \[ Y(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k} \]

- I.E., It’s a discrete version of the Laplace:
  \[ f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{Z}{z - e^{-aT}} \]

The z-transform

- In practice, you’ll use look-up tables or computer tools (ie. Matlab) to find the z-transform of your functions

<table>
<thead>
<tr>
<th>( F(s) )</th>
<th>( F(kt) )</th>
<th>( F(x) )</th>
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<tbody>
<tr>
<td>( \frac{1}{s} )</td>
<td>1</td>
<td>( \frac{z}{z - 1} )</td>
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<tr>
<td>( \frac{1}{s^2} )</td>
<td>( kT )</td>
<td>( \frac{Tz}{(z - 1)^2} )</td>
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<tr>
<td>( \frac{1}{s + a} )</td>
<td>( e^{-akT} )</td>
<td>( \frac{z}{z - e^{-aT}} )</td>
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<tr>
<td>( \frac{1}{(s + a)^2} )</td>
<td>( kTe^{-akT} )</td>
<td>( \frac{zTe^{-aT}}{(z - e^{-aT})^2} )</td>
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<tr>
<td>( \frac{1}{s^2 + a^2} )</td>
<td>( \sin(akT) )</td>
<td>( \frac{z \sin aT}{z^2 - (2 \cos aT)z + 1} )</td>
</tr>
</tbody>
</table>
An example!

- Back to our difference equation:
  \[ y(k) = x(k) + Ax(k - 1) - By(k - 1) \]

becomes

\[
Y(z) = X(z) + Az^{-1}X(z) - Bz^{-1}Y(z)
\]

\[(z + B)Y(z) = (z + A)X(z)\]

which yields the transfer function:

\[
\frac{Y(z)}{X(z)} = \frac{z + A}{z + B}
\]

Note: It is also not uncommon to see systems expressed as polynomials in \(z^{-n}\)

This looks familiar…

- Compare:

\[
\frac{Y(s)}{X(s)} = \frac{s + 2}{s + 1} \quad VS \quad \frac{Y(z)}{X(z)} = \frac{z + A}{z + B}
\]

How are the Laplace and \(z\) domain representations related?
Deep insight #1

The mapping between continuous and discrete poles and zeros acts like a distortion of the plane

\[ \text{Im}(z) \quad \text{Re}(z) \]

\[ \text{Im}(s) \quad \text{Re}(s) \]

1 max frequency

Region of Convergence

- For the convergence of \( X(z) \) we require that

\[ \sum \left| a z^{-1} \right| < \infty \]

- Thus, the ROC is the range of values of \( z \) for which \( |a z^{-1}| < 1 \) or, equivalently, \( |z| > |a| \). Then

\[ X(z) = \frac{z}{z - a} \quad |z| > |a| \]
Z-Transform Properties: Linearity

Linearity:

\[ a_1 y_1[n] + a_2 y_2[n] \xrightarrow{Z} a_1 Y_1(z) + a_2 Y_2(z) \]

Z-Transform Properties: Time Shifting

- Two Special Cases:
  - \( z^{-1} \): the unit-delay operator:
    \[ x[n - 1] \leftrightarrow z^{-1} X(z) \quad R' = R \cap \{0 < |z|\} \]
  - \( z \): unit-advance operator:
    \[ x[n + 1] \leftrightarrow z X(z) \quad R' = R \cap \{|z| < \infty\} \]
Z-Transform Properties

- Time Reversal
  \[ x[n] \leftrightarrow X(z) \quad \text{ROC} = R \]
  \[ x[-n] \leftrightarrow X \left( \frac{1}{z} \right) \quad R' = \frac{1}{R} \]

- Multiplication by \( z^n \)
  \[ x[n] \leftrightarrow X(z) \quad \text{ROC} = R \]
  \[ z[n] \leftrightarrow X \left( \frac{z}{z_0} \right) \quad R' = \frac{|z_0|}{R} \]

- Multiplication by \( n \) (or Differentiation in \( z \)):
  \[ x[n] \leftrightarrow X(z) \quad \text{ROC} = R \]
  \[ n x[n] \leftrightarrow -\frac{dX(z)}{dz} \quad R' = R \]

- Convolution
  \[ x_1[n] \ast x_2[n] \leftrightarrow X_1(z) X_2(z) \quad R' = R_1 \cap R_2 \]

---

z-transform & Systems: Damping and natural frequency

\[ z = e^{sT} \quad \text{where} \quad s = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2} \]

[Adapted from Franklin, Powell and Emami-Naeini]
In general: $z$-plane stability

- In the $z$-domain, the unit circle is the system stability bound

Frequency

- How often the signal repeats
- Can be analyzed through Fourier Transform

Examples:

\[ \mathcal{F} \]
Noise

Various Types:
- Thermal (white):
  - Johnson noise, from thermal energy inherent in mass.

- Flicker or 1/f noise:
  - Pink noise
  - More noise at lower frequency

- Shot noise:
  - Noise from quantum effects as current flows across a semiconductor barrier

- Avalanche noise:
  - Noise from junction at breakdown (circuit at discharge)
How to beat the noise

- Filtering (Narrow-banding): Only look at particular portion of frequency space
- Multiple measurements …
- Other (modulation, etc.) …

Noise $\subseteq$ Uncertainty

- **Uncertainty**: All measurement has some approximation
  A. **Statistical uncertainty**: quantified by mean & variance
  B. **Systematic uncertainty**: non-random error sources

- **Law of Propagation of Uncertainty**
  - Combined uncertainty is root squared

\[ u_C = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} \]
### Treating Uncertainty with Multiple Measurements

1. **Over time**: multiple readings of a quantity over time  
   - “stationary” or “ergodic” system  
   - Sometimes called “integrating”  

2. **Over space**: single measurement (summed) from multiple sensors each distributed in space  

3. **Same Measurand**: multiple measurements take of the same observable quantity by multiple, related instruments  
   - e.g., measure position & velocity simultaneously  
   - Basic “sensor fusion”

\[
\sigma_{\text{final}} = \left[\frac{1}{\sigma_1^{-1} + \frac{1}{\sigma_2^{-1}} + \cdots + \frac{1}{\sigma_n^{-1}}}\right]^{-1}
\]

---

### Multiple Measurements Example

- What time was it when this picture was taken?  
- What was the temperature in the room?
Filters

Specified Values:

- $G_p = \text{minimum passband gain}$

Typically:

$$G_p = \frac{1}{\sqrt{2}} = -3dB$$

- $G_s = \text{maximum stopband gain}$

  - Low, not zero (sorry!)
  - For realizable filters, the gain cannot be zero over a finite band (Paley-Wiener condition)

- **Transition Band:**

  transition from the passband to the stopband $\Rightarrow \omega_p \neq \omega_s$
Announcements:

• Assignment 1 Solutions:
  – Extended to Friday

• Lab 2 (Experiment 3)
  ➔ We are rerunning it again this week 😊
  (for those who want to do it “post-theory”)

• Lab 3 (Experiment 4)
  – Will run on Week 9!
  ∴ Week 8 has the ANZAC holiday

Next Time in Linear Systems ….

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Frequency Response of a LTI System

\[ H(s) \]

Input: \( U(s) \)
Output: \( Y(s) = H(s) \cdot U(s) \)

Example:
\[ H(s) = \frac{s + 0.1}{s + 0.5} = \frac{s + 0.1}{s + 5} \]

\[ s = j\omega \]

\[ H(j\omega) = \frac{j\omega + 0.1}{j\omega + 5} \]

\[ |H(j\omega)| = \frac{\sqrt{\omega^2 + 0.01}}{\sqrt{\omega^2 + 25}} \]

\[ \angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right) \]

Why?
\[ H(j\omega) = \frac{j\omega + A}{j\omega + B} \]

\[ H(j\omega) = \frac{\sqrt{\omega^2 + A^2}}{\sqrt{\omega^2 + B^2}} \]

\[ \angle H(j\omega) = \tan^{-1}\left(\frac{A}{\omega}\right) - \tan^{-1}\left(\frac{B}{\omega}\right) \]
Example 2: Time Delay

\[
H(s) = e^{-st}
\]

\[
H(j\omega) = e^{-j\omega t}
\]

\[
|H(j\omega)| = 1 \quad \text{and} \quad \angle H(j\omega) = -\omega t
\]

--- FILTERS ---

--- Colored Signal ---

\[
\rightarrow \Rightarrow G
\]

\[
\omega
\]