### Today:

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- New! Revised order
Announcements:

- Assignment 1 Solutions:
  - Questions 1-5:
    - “all or nothing”
    - Okay… **0 or 3 or 5 points**
      (+ 1-2 bonus for something truly exceptional)
  - Lab 2 (Experiment 3)
    - Is to give you a “feeling” / “intuition” for digital systems
    - Feels a little “ahead” of it’s time
    ➔ We are rerunning it again next week 😊
      (for those who want to do it “post-theory”)
- Lab 3 (Experiment 4)
  - **Will run on Week 9!**
    ∴ Week 8 has the ANZAC holiday

Refresher:
Aliasing & Sampling

- Nyquist:

\[ f_h < \frac{f_r}{2} \]

- Spectral Folding:

\[ f_{image}(N) = f - N f_s \]
Quick Background:
Pole-Zero Diagrams & The Root Locus

- The transfer function for a closed-loop system can be easily calculated:

\[ y = CH(r - y) \]
\[ y + CHy = CHr \]
\[ y = \frac{CH}{r} \]
\[ r = \frac{1}{1 + CH} \]

\[ r \]
\[ + \]
\[ e \]
\[ C \]
controller

\[ u \]
\[ H \]
plant

\[ y \]

Quick Background:
Pole-Zero Diagrams & The Root Locus

- We often care about the effect of increasing gain of a control compensator design:

\[ \frac{y}{r} = \frac{kCH}{1 + kCH} \]

Multiplying by denominator:

\[ \frac{y}{r} = \frac{kC_nH_n}{C_dH_d + kC_nH_n} \]

characteristic polynomial

\[ r \]
\[ + \]
\[ e \]
\[ k \]

\[ C \]

\[ u \]
\[ H \]

\[ y \]
Quick Background:
Pole-Zero Diagrams & The Root Locus

• Pole positions change with increasing gain
  – The trajectory of closed-loop poles on the pole-zero plot with changing $k$ is called the “root locus”
  – This is sometimes quite complex

(Copied with these with computers)

Coping with Complexity

• Transfer functions help control complexity
  – Recall the Laplace transform:
    \[ \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt = F(s) \]
    where \[ \mathcal{L}\{f(t)\} = sF(s) \]
    \[ x(t) \xrightarrow{H(s)} y(t) \]

Is there a something similar for sampled systems?
The $z$-transform

- The discrete equivalent is the $z$-Transform$^\dagger$:

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)z^{-k} = F(z)$$

and

$$Z\{f(k-1)\} = z^{-1}F(z)$$

Convenient!

$^\dagger$This is not an approximation, but approximations are easier to derive

---

What about the Discrete Domain? [Lecture 4-Slide 10]
The \( z \)-transform

- Some useful properties
  - **Delay by \( n \) samples**: \( Z \{ f(k - n) \} = z^{-n}F(z) \)
  - **Linear**: \( Z \{ af(k) + bg(k) \} = aF(z) + bG(z) \)
  - **Convolution**: \( Z \{ f(k) * g(k) \} = F(z)G(z) \)

So, all those block diagram manipulation tools you know and love will work just the same!

<table>
<thead>
<tr>
<th>( F(s) )</th>
<th>( F(k) )</th>
<th>( F(z) )</th>
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</thead>
<tbody>
<tr>
<td>( \frac{1}{s} )</td>
<td>1</td>
<td>( \frac{z}{z - 1} )</td>
</tr>
<tr>
<td>( \frac{1}{s^2} )</td>
<td>( kT )</td>
<td>( \frac{Tz}{(z - 1)^2} )</td>
</tr>
<tr>
<td>( \frac{1}{s + a} )</td>
<td>( e^{-akT} )</td>
<td>( \frac{z}{z - e^{-aT}} )</td>
</tr>
<tr>
<td>( \frac{1}{(s + a)^2} )</td>
<td>( kTe^{-akT} )</td>
<td>( \frac{zT e^{-aT}}{(z - e^{-aT})^2} )</td>
</tr>
<tr>
<td>( \frac{1}{s^2 + a^2} )</td>
<td>( \sin(akT) )</td>
<td>( \frac{z \sin aT}{z^2 - (2 \cos aT)z + 1} )</td>
</tr>
</tbody>
</table>
Final value theorem

- An important question: what is the steady-state output a stable system at $t = \infty$?
  - For continuous systems, this is found by:
    $$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$
  - The discrete equivalent is:
    $$\lim_{k \to \infty} x(k) = \lim_{z \to 1} (1 - z^{-1})X(z)$$
  - (Provided the system is stable)

An example!

- Back to our difference equation:
  $$y(k) = x(k) + Ax(k - 1) - By(k - 1)$$
becomes
  $$Y(z) = X(z) + Az^{-1}X(z) - Bz^{-1}Y(z)$$
  $$(z + B)Y(z) = (z + A)X(z)$$
which yields the transfer function:
  $$\frac{Y(z)}{X(z)} = \frac{z + A}{z + B}$$

Note: It is also not uncommon to see systems expressed as polynomials in $z^{-n}$
This looks familiar…

- Compare:

\[
\frac{Y(s)}{X(s)} = \frac{s+2}{s+1} \quad \text{vs} \quad \frac{Y(z)}{X(z)} = \frac{z+A}{z+B}
\]

How are the Laplace and z domain representations related?

Consider the simplest system

- Take a first-order response:

\[
f(t) = e^{-at} \Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s + a}
\]

- The discrete version is:

\[
f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{z}{z - e^{-aT}}
\]

The equivalent system poles are related by

\[
z = e^{sT}
\]

That sounds somewhat profound… but what does it mean?
The $z$-Plane

- $z$-domain poles and zeros can be plotted just like $s$-domain poles and zeros:

Deep insight #1

The mapping between continuous and discrete poles and zeros acts like a distortion of the plane
The $z$-plane

- We can understand system response by pole location in the $z$-plane

[Adapted from Franklin, Powell and Emami-Naeini]

Effect of pole positions

- We can understand system response by pole location in the $z$-plane

Most like the $s$-plane
Effect of pole positions

- We can understand system response by pole location in the $z$-plane.

Increasing frequency

![Diagram showing the effect of pole positions on system response.](attachment:image.png)
Damping and natural frequency

\[ z = e^{st} \text{ where } s = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2} \]

\[ Re(z) \]
\[ Img(z) \]
\[ \zeta = 0 \]
\[ \zeta = 1 \]

\[ \omega_n = \frac{\pi}{T} \]

\[ z \text{-plane stability} \]
• In the \( z \)-domain, the unit circle is the system stability bound

\[ Re(s) \]
\[ Img(s) \]
**z-plane stability**

- In the $z$-domain, the unit circle is the system stability bound.

![Diagram showing stability in the $z$-plane]

**z-plane stability**

- The $z$-plane root-locus in closed loop feedback behaves just like the $s$-plane.

![Diagram showing stability in the $s$-plane]
Recall dynamic responses

- Moving pole positions change system response characteristics

Recall dynamic responses

- Ditto the z-plane:
Deep insight #2

- Gains that stabilise continuous systems can actually **destabilise** digital systems!