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ELEC 3004: Systems: Signals & Controls
Dr. Surya Singh
Lecture 10
Cool Signal Share: Eulerian Video Magnification for Revealing Subtle Changes in the World

Eulerian Video Magnification for Revealing Subtle Changes in the World

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Announcements:

- Assignment 1 Solutions:
  - Posted
  - Please try to get the peer review marks in by April 16 at 11:59pm
Signals Processing: Seeing The Light

Think of a Signal Processing like a prism:
“Destructs a source signal into its constituent frequencies”

Fourier Series
• Any finite power, periodic, signal $x(t)$
  – period $T$
• can be represented as ($\infty$) summation of
  – sine and cosine waves
• Called: Trigonometrical Fourier Series

\[
x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nw_0 t) + B_n \sin(nw_0 t)
\]

• Fundamental frequency $w_0 = 2\pi/T$ rad/s or $1/T$ Hz
• DC (average) value $A_0/2$
• Signal measured (or known) as a function of an independent variable
  – e.g., time: \( y = f(t) \)
• However, this independent variable may not be the most appropriate/informative
  – e.g., frequency: \( Y = f(w) \)
• Therefore, need to transform from one domain to the other
  – e.g., time \( \Leftrightarrow \) frequency
  – As used by the human ear (and eye)

### Signal processing uses Fourier, Laplace, & z transforms etc

![Time-domain and frequency-domain graphs]

**Frequency representation (spectrum) shows signal contains:**
- 2Hz and 5Hz components (sinewaves) of equal amplitude
Fourier Series Coefficients

- \( A_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(nw_0 t) \, dt \quad n = 0, 1, 2, \ldots \)

- \( B_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(nw_0 t) \, dt \quad n = 1, 2, 3, \ldots \)

Note: Limits of integration can vary, provided they cover one period.
Fourier Series Coefficients

- Approximation with 1st, 3rd, 5th, & 7th Harmonics added, note:
  - ‘Ringing’ on edges due to series truncation
  - Often referred to as Gibb’s phenomenon
- Fourier series converges to original signal if
  - Dirichlet conditions satisfied
  - Closer approximation with more harmonics
Dirichlet Conditions

- For Fourier series to converge, $f(t)$ must be defined & single valued
- Continuous and have a finite number of finite discontinuities within a periodic interval, and
- Piecewise continuous in periodic interval, as must $f'(t)$

Example: Square wave

\[
x(t) = \begin{cases} 
1, & 0 < t < 1; \\
-1, & 1 < t < 2; \\
 x(t + 2). & \text{periodic! i.e., } x(t + 2) = x(t)
\end{cases}
\]

\[
A_n = \int_0^1 x(t) \cos(n \pi t) dt = \frac{1}{n \pi} \cos(n \pi) \left[ \int_0^1 \cos(n \pi) dt - \int_1^2 \cos(n \pi) dt \right] 
\]

\[
A_n = \left[ \frac{-\sin(n \pi)}{n \pi} \right]_0^1 - \left[ \frac{-\sin(n \pi)}{n \pi} \right]_1^2 = 0 
\]

No cos terms as $\sin(n \pi) = 0 \forall n$

$x(t)$ has odd symmetry

\[
B_n = \frac{1}{n \pi} \int_0^1 x(t) \sin(n \pi t) dt = \frac{1}{n \pi} \sin(n \pi) \left[ \int_0^1 \sin(n \pi) dt - \int_1^2 \sin(n \pi) dt \right] 
\]

\[
B_n = \left[ \frac{-\cos(n \pi)}{n \pi} \right]_0^1 - \left[ \frac{-\cos(n \pi)}{n \pi} \right]_1^2 = -\cos(n \pi) + \frac{1}{n \pi} + \frac{1}{n \pi} \cos(n \pi) 
\]

$B_n = \frac{2}{n \pi} (1 - \cos(n \pi))$ Sin terms only
Example: Square wave

Therefore, Trigonometric Fourier series is,

\[ x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - \cos(n\pi)) \sin(n\pi) \]

Expanding the terms gives,

\[
x(t) = \frac{4}{\pi} \sin(\pi t) \quad \text{fundamental} \\
+ 0 \quad \text{(second harmonic)} \\
+ \frac{4}{3\pi} \sin(3\pi t) \quad \text{(third harmonic)} \\
+ 0 \quad \text{(fourth harmonic)} \\
+ \frac{4}{5\pi} \sin(5\pi t) \quad \text{(fifth harmonic)} \\
+ \text{etc}
\]

- Only odd harmonics;
- In proportion 1, 1/3, 1/5, 1/7,…
- Higher harmonics contribute less;
- Therefore, converges

Complex Fourier Series (CFS)

- Also called Exponential Fourier series
- FS as a Complex phasor summation
  
- As it uses Euler’s relation

\[
A \exp(jw_0t) = A \cos(w_0t) + jA \sin(w_0t)
\]

which implies,

\[
\cos(nw_0t) = \frac{\exp(jnw_0t) + \exp(-jnw_0t)}{2}
\]

\[
\sin(nw_0t) = \frac{\exp(jnw_0t) - \exp(-jnw_0t)}{2j}
\]

\[
x(t) = \sum_{n=-\infty}^{\infty} X_n \exp(jnw_0t)
\]

Where \(X_n\) are the CFS coefficients
Complex Phasor Summation

Complex Fourier Coefficients

- Again, $X_n$ calculated from $x(t)$
- Only one set of coefficients, $X_n$
  - but, generally they are complex

$$X_n = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \exp(-jnw_0 t) dt$$

Remember: fundamental $w_0 = 2\pi/T$!
Relationships

- There is a simple relationship between
  - trigonometrical and
  - complex Fourier coefficients,

\[
X_0 = \frac{A_0}{2}
\]

\[
X_n = \begin{cases} 
A_n - jB_n, & n > 0; \\
\frac{2}{A_n + jB_n}, & n < 0.
\end{cases}
\]

Constrained to be symmetrical, i.e., complex conjugate

\[X_{-n} = X_n^*\]

Therefore, can calculate simplest form and convert

Example: Complex FS

- Consider the pulse train signal

\[
x(t) = \begin{cases} 
A, & 0 \leq |t| \leq \frac{T}{2}; \\
0, & \frac{T}{2} < |t| \leq T; \\
x(t+T).
\end{cases}
\]

- Has complex Fourier series:

\[
X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \exp(-jn\omega_0 t) dt = \frac{1}{T} \int_{-T/2}^{T/2} A \exp(-jn\omega_0 t) dt
\]

\[
= \frac{-A \tau}{j n \omega_0 T \tau} \left[ \exp\left(-jn\omega_0 \frac{\tau}{2}\right) - \exp\left(jn\omega_0 \frac{\tau}{2}\right) \right]
\]

Note: \( n \) is the ind. variable
Example: Complex FS

• Which using Euler’s identity reduces to:

\[ X_n = \frac{A \tau \sin(nw_0 \tau/2)}{T} = \frac{A \tau}{T} \sin(nw_0 \tau/2) \]

\[ w_0 = \frac{2\pi}{T} \]

Note: letting \( \theta = \frac{n\omega_0 \tau}{2} \)

\[ \exp(-j\theta) - \exp(j\theta) \]

\[ = \cos(-\theta) + j \sin(-\theta) - (\cos(\theta) + j \sin(\theta)) \]

\[ = \cos(\theta) - j \sin(\theta) - \cos(\theta) - j \sin(\theta) = -2j \sin(\theta) \]

Note: complex conj symmetry

\[ \pm \pi \rightarrow \text{ve real value} \]

Note: complex conj symmetry
Duty Cycle $\tau/T = 1/4$

Amplitude, $|X_n|$

Phase, $\angle(X_n)$

Duty Cycle $\tau/T = 1/8$

Amplitude, $|X_n|$

Phase, $\angle(X_n)$

Coefficients getting smaller!

$4^{th}$, $8^{th}$, $12^{th}$, $\ldots = 0$

$8^{th}$, $16^{th}$, $\ldots = 0$
Complex FS of Pulse Train

- Amplitude spectrum has ‘sinc’ (‘sa’) envelope
  - Amplitude reduces as duty cycle decreases
  - DC coefficient X0 (n=0) present
    - compare to first example
  - Duty cycle $\tau/T$: XM$\tau/T = 0$ (M = 1,2,3,...)
  - Even symmetry: $|X-n| = |X_n|
    - For real (not complex) signals (all we shall consider)
    - Often only plot positive frequency, e.g., Matlab

- Phase spectrum
  - Odd symmetry: $\angle X-n = -\angle X_n$

**Complex conjugate (Hermitian) symmetry is general property for real $x(t)$**
Interpretation of Fourier Series

- Represents periodic signals (T = 2π/w₀)
  - as sum of cosine waves: “cosine series”
    - at harmonic frequencies 0, w₀, 2w₀, 3w₀, …
    - |Xₙ| is half amplitude of nth harmonic
    - ∠Xₙ is phase shift of nth harmonic
- Distribution with harmonic number of
  - both amplitude & phase
  - called a frequency spectrum (discrete)

\[ x(t) = X₀ + \sum_{n=1}^{\infty} 2 |Xₙ| \cos(nw₀t + ∠Xₙ) \]

i.e., Harmonics only

Orthogonal Expansions

- Trigonometrical and Complex FS both
  - Orthogonal expansions
- Because harmonically related
  - sines, cosines, and complex phasors are all orthogonal
    - Product of f₁(t) & f₂(t) integrates to zero over 1 period

\[ \frac{1}{T} \int_{-T/2}^{T/2} \exp(jnw₀t) \exp^*(jmw₀t)dt = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{if } m \neq n. \end{cases} \]

This means we can calculate each Xₙ independently
(instead of solving n simultaneous equations)
Example: Calculate Power in $x(t)$

A simple periodic signal $x(t)$ where:

$$x(t) = a_1 \sin(\omega_0 t) + a_2 \sin(2\omega_0 t)$$

The power:

$$P = \frac{1}{T} \int_0^T x^2(t) \, dt$$

$$= \frac{1}{T} \int_0^T a_1^2 \sin^2(\omega_0 t) \, dt + \frac{1}{T} \int_0^T a_2^2 \sin^2(2\omega_0 t) \, dt \quad \text{Orthogonal} \rightarrow 0$$

$$= \frac{1}{T} \int_0^T \frac{1}{2} a_1^2 \sin^2(\omega_0 t) \, dt + \frac{1}{T} \int_0^T \frac{1}{2} a_2^2 \sin^2(2\omega_0 t) \, dt$$

$$= \frac{a_1^2}{2} + \frac{a_2^2}{2}$$

Total = sum of power in 2 sine waves

Parseval’s Theorem

- Direct consequence of orthogonality
- Calculate power of a signal either
  - in time domain, i.e., from $x(t)$, or
  - in frequency domain, i.e., from $X_n$

$$P = \frac{1}{T} \int_0^T x^2(t) \, dt = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Continuous & periodic

Discrete
Duty Cycle $\tau/T = 1/2$

Normalised Amplitude, $|X_n|*T/\tau$

Angular frequency (w)

Phase, angle($X_n$)

Angular frequency (w)

Note change of axes!
Duty Cycle \( \frac{\tau}{T} = 1/8 \)

Normalised Amplitude, \( \frac{|X_n| T}{\tau} \)

Angular frequency (\( w \))

Phase, \( \angle(X_n) \)

Angular frequency (\( w \))
Fourier Transform

- Fourier series
  - Only applicable to periodic signals
- Real world signals are rarely periodic
- Develop Fourier transform by
  - Examining a periodic signal
  - Extending the period to infinity

Fourier Transform

- Problem: as $T \to \infty$, $X_n \to 0$
  - i.e., Fourier coefficients vanish!
- Solution: re-define coefficients
  - $X'_n = T \times X_n$
- As $T \to \infty$
  - (harmonic frequency) $nw_0 \to w$ (continuous freq.)
  - (discrete spectrum) $X'_n \to X(w)$ (continuous spect.)
  - $w_0$ (fundamental freq.) reduces $\to dw$ (differential)
    - Summation becomes integration
Fourier Transform

Note missing 1/T

Modified Fourier series

\[ X'_n = \int_{-T/2}^{+T/2} x(t) \exp(-jnw_0t) dt \]

\[ X(w) = \lim_{T \to \infty} \int_{-T/2}^{+T/2} x(t) \exp(-jnw_0t) dt \]

Note: as \( T \to \infty \)

\[ X(w) = \int_{-\infty}^{\infty} x(t) \exp(-jwt) dt \]

\[ x(t) = \lim_{T \to \infty} \sum_{n=-\infty}^{\infty} X'_n \exp(jnw_0t) \frac{W_0}{2\pi} \]

Inverse Fourier series

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) \exp(jwt) dw \]

Inverse Fourier transform

\[ x(t) = F^{-1}\{X(w)\} \]

As a example of the evaluation of the Fourier transform, consider the finite energy signal \( x(t) \) illustrated in Figure 1.8.

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) dt \]

\[ = \int_{-\pi/2}^{\pi/2} x(t) \exp(-j\omega t) dt \]

\[ = \frac{-A}{j\omega} \left[ \exp\left(\frac{-j\omega \tau}{2}\right) - \exp\left(\frac{j\omega \tau}{2}\right) \right] \]

Using Euler’s relation

\[ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2j} \]

Note the similarity to Previous FS examples

\[ \sin(\omega \tau/2) \]

\[ = A\tau \sin(\omega \tau/2) \]

\[ = A\tau \text{sinc}(\omega \tau/2) \]

Figure 1.8 Fourier transform of a rectangular pulse.
Summary!