METR4202 -- Robotics
Tutorial 5 – Week 5: Trajectory Generation & Motion Planning

Reading
Please read/review chapter 9 of Robotics, Vision and Control.

Questions
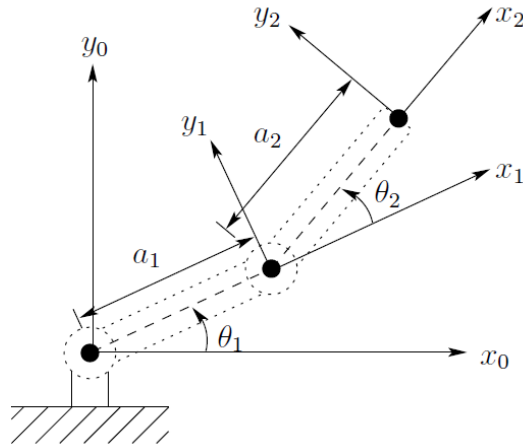

Figure 1: Two DOF Robot manipulator

1. Write the full equation of motion for the 2R arm above
   (i.e., τ_1 and τ_2 as a function of θ_1 and θ_2 and its derivatives)

Start with the masses of links: m_1 and m_2, to get the Mass Matrix recall (lecture 5)

$$M = \sum_{i=1}^{N} \left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i} \right)$$

$$M = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T I_1 J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T I_2 J_{\omega_2}$$

Note that:

m_i = the mass of the ith link

m_{ij} = the ij element of the mass matrix

$$m_{ijk} \equiv \frac{\partial m_{ij}}{\partial q_k}$$

Note this is with respect to the configuration variable, not time.

On that subject, the derivative with respect to time would be: $\dfrac{d}{dt} m_{ij} = \sum_{k=1}^{N} m_{ijk} \dot{q}_k$

The center of mass of each link is at the joint center, this $l_1 \equiv a_1/2$ and $l_2 \equiv a_2/2$

To compute the Jacobians (J_v and J_ω), we need to calculate the forward kinematics.

Recall that the position vectors (Lec 3, Slide 34) for a 2R arm are:

$$^{0}P_{1} = \begin{bmatrix} a_1 C_1 \\ a_1 S_1 \\ 0 \end{bmatrix} \text{ (this reads as "Position of Frame 1 as seen in 0"),} \quad ^{0}P_{2} = \begin{bmatrix} a_1 C_1 + a_2 C_{12} \\ a_1 S_2 + a_2 S_{12} \\ 0 \end{bmatrix}$$

Thus with respect to Frame {0}, the translational velocity Jacobians (i.e., the matrices that encode the differential relationship between joint velocities and workspace tip velocities) are found by direct differentiation of the position vectors $^{0}P_1$ and $^{0}P_2$.

$$^{0}J_{v_1} = \begin{bmatrix} -a_1 S_1 & 0 \\ a_1 C_1 & 0 \\ 0 & 0 \end{bmatrix}, ^{0}J_{v_2} = \begin{bmatrix} -a_1 S_1 - a_1 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow m_1 J_{v_1}^{T} J_{v_1} = \begin{bmatrix} m_1 a_1^2 & 0 \\ 0 & 0 \end{bmatrix}, m_2 J_{v_2}^{T} J_{v_2} = \begin{bmatrix} m_2 \left(a_1^2 + a_2^2 + 2a_1 a_2 C_2 \right) & m_2 \left(a_2^2 + a_1 a_2 C_2 \right) \\ m_2 \left(a_2^2 + a_1 a_2 C_2 \right) & m_2 a_2^2 \end{bmatrix}$$

The rotational velocity Jacobian matrices with respect to Frame {0} are given by

$$J_{\omega_1} = \begin{bmatrix} \bar{\varepsilon}_1 \mathbf{z}_1 & \mathbf{0} \end{bmatrix}, J_{\omega_2} = \begin{bmatrix} \bar{\varepsilon}_1 \mathbf{z}_1 & \bar{\varepsilon}_2 \mathbf{z}_2 \end{bmatrix}$$

As both joints are revolute (ε=0), these matrices are $J_{\omega_1} = \begin{bmatrix} \mathbf{z}_1 & \mathbf{0} \end{bmatrix}, J_{\omega_2} = \begin{bmatrix} \mathbf{z}_1 & \mathbf{z}_2 \end{bmatrix}$

Thus, $J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$, and after some substitution and simplification we have

$$J_{\omega_1}^{T} I_1 J_{\omega_1} = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, J_{\omega_2}^{T} I_2 J_{\omega_2} = \begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} \text{ where I is about the } z\text{-axis } (I_1 = I_{\{zz\}1} \text{ and } I_2 = I_{\{zz\}2})$$

Finally, the mass matrix, M is

$$M = \begin{bmatrix} m_1 a_1^2 + I_1 + m_2 \left(a_1^2 + a_2^2 + 2a_1 a_2 C_2 \right) + I_2 & m_2 \left(a_2^2 + a_1 a_2 C_2 \right) + I_2 \\ m_2 \left(a_2^2 + a_1 a_2 C_2 \right) + I_2 & m_2 a_2^2 + I_2 \end{bmatrix}$$

The Centrifugal and Coriolis Matrix **v** is found directly by recalling Christoffel symbols (please review Christoffel symbols from dynamics and the mass notation from the previous page)

$$b_{i,jk} = \frac{1}{2} \left(m_{ijk} + m_{ikj} - m_{jki} \right) \text{ and with } b_{iii} = b_{iji} = 0,$$

the Centrifugal matrix becomes

$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} m_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \left(\frac{\partial m_{11}}{\partial \theta_2} \right) \\ 0 \end{bmatrix} = \begin{bmatrix} -m_2 a_1 a_2 S_2 \\ 0 \end{bmatrix},$$

and the Coriolis matrix can be written as

$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{122} \\ -\frac{1}{2} m_{112} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \left(\frac{\partial m_{12}}{\partial \theta_2} \right) \\ -\frac{1}{2} \left(\frac{\partial m_{11}}{\partial \theta_2} \right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -m_2 a_1 a_2 S_2 \\ m_2 a_1 a_2 S_2 & 0 \end{bmatrix}$$

Summing this together gives

$$V = \begin{bmatrix} -m_2 a_1 a_2 S_2 \\ 0 \end{bmatrix} \left(\dot{\theta}_1 \dot{\theta}_2 \right) + \begin{bmatrix} 0 & -m_2 a_1 a_2 S_2 \\ m_2 a_1 a_2 S_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix}$$

The next factor to consider is gravity.

While the problem does not specify a gravity direction, we assume it is acting parallel to the y-axis. This gives $\mathbf{g} = \begin{bmatrix} 0 & -g & 0 \end{bmatrix}$. (Note that if we later wish to assume that gravity is acting along the z-axis (into the page), this could be treated by setting $\mathbf{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}$)

With respect to Frame {0}, the gravity vector can be calculated as

$$\mathbf{G} = -\left[J_{v_{C1}}^T m_{C1} \mathbf{g} + J_{v_{C2}}^T m_{C2} \mathbf{g} \right]$$

However, we have to be careful because the gravity acts at the mass center (which is represented by the notation C1 and C2). Again, recall that we have $l_1 = a_1/2$ and $l_2 = a_2/2$. Given the structure of the problem, the Jacbobians are be determined by inspection. Thus,

$$^0G = -\begin{bmatrix} -l_1 S_1 & l_1 C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -a_1 S_1 - l_2 S_{12} & a_1 C_1 + l_2 C_{12} & 0 \\ -l_2 S_{12} & l_2 C_{12} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

$$^0G = \begin{bmatrix} (m_1 l_1 + m_2 a_1) C_1 + m_2 l_2 C_{12} \\ m_2 l_2 C_{12} \end{bmatrix}(g) = \begin{bmatrix} \left(\tfrac{1}{2} m_1 + m_2\right) a_1 C_1 + \tfrac{1}{2} m_2 a_2 C_{12} \\ \tfrac{1}{2} m_2 a_2 C_{12} \end{bmatrix}(g)$$

The **Equations of Motion** can be found by putting these terms together to give
(for review see also Lecture 4, Slide 30 and Lecture 5, Slide 7)

$$\boldsymbol{\tau} = M(\theta)\ddot{\boldsymbol{\theta}} + \mathbf{v}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{g}(\boldsymbol{\theta})$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1 a_1^2 + I_1 + m_2\left(a_1^2 + a_2^2 + 2a_1 a_2 C_2\right) + I_2 & m_2\left(a_2^2 + a_1 a_2 C_2\right) + I_2 \\ m_2\left(a_2^2 + a_1 a_2 C_2\right) + I_2 & m_2 a_2^2 + I_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \left(\dot{\theta}_1 \dot{\theta}_2\right)\begin{bmatrix} -m_2 a_1 a_2 S_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -m_2 a_1 a_2 S_2 \\ m_2 a_1 a_2 S_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + (g)\begin{bmatrix} \left(\tfrac{1}{2} m_1 + m_2\right) a_1 C_1 + \tfrac{1}{2} m_2 a_2 C_{12} \\ \tfrac{1}{2} m_2 a_2 C_{12} \end{bmatrix}$$

Challenge Question:

Inverse Kinematics & Trajectory Generation

    A small humanoid robot is being programmed to place a hat on its head. The objective is to place the hat in the position shown by the dashed outline in the figure below. Assume that the arm is composed of 3 revolute joints and is constrained to move in the plane of the page. The arm consists of 3 links with dimensions: $L_1=0.4$, $L_2=0.3$, $L_3=0.1$.

    In order to place the hat on its head, assume that we must place the edge of the hat brim at a location 0.5m above its shoulder joint. The hat brim should be in a horizontal position and is gripped at its edge by the hand and is aligned with the last link of the arm. Please calculate/plot valid workspace (e.g., from the frame located at the right-most end of the brim where the robot is grasping it) and joint trajectories to place the hat correctly.